V<sup>0</sup> as a Tarski universe 00000  $\mathcal{V}^0$  as a precategory and its Rezk completion 0000

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# The category of iterative sets in HoTT

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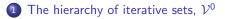
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## Outline







(3)  $\mathcal{V}^0$  as a precategory and its Rezk completion

V<sup>0</sup> as a Tarski universe

 $\mathcal{V}^0$  as a precategory and its Rezk completion

## The hierarchy of iterative sets

Recall the hierarchy of iterative sets:



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### Definition (The hierarchy of iterative sets)

We define the following predicate on Aczel's type  $W_{A:U}A:$ 

iterative-set :  $W_{A:\mathcal{U}}A \rightarrow \mathsf{Type}$ 

iterative-set (sup A f) := is-embedding  $f \times \prod_{a:A}$  iterative-set (f a)

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and take the corresponding subtype of  $W_{A:U}A$ :

$$\mathcal{V}^0 := \sum_{x: W_{A:\mathcal{U}}A} \text{iterative-set } x$$

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V<sup>0</sup> as a Tarski universe

# $\mathcal{V}^0$ is an h-set

#### Theorem

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#### Theorem

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#### Proof.

Sketch:

- $\mathcal{V}^0$  is a fixpoint of the functor  $X \mapsto \sum_{A:U} (A \hookrightarrow X)$
- For  $(A, f), (B, g) : \sum_{A:U} (A \hookrightarrow \mathcal{V}^0)$  we have:

$$((A, f) = (B, g)) \simeq \prod_{x:\mathcal{V}^0} (\text{fiber } f \ x \simeq \text{fiber } g \ x)$$

• All the fibers are propositions

 $\mathcal{V}^0$  as a Tarski universe

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# $\mathcal{V}^0$ as a Tarski universe

A Tarski universe is a type with a type family.



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### A Tarski universe is a type with a type family.

#### Definition

We define the following decoding function:

 $\mathsf{EI}: \mathcal{V}^0 \to \mathcal{U}$  $\mathsf{EI} (\sup A f, p) := A$ 

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# Types and type formers in $\mathcal{V}^{0}$

We have the following in  $\mathcal{V}^0$  if we have them in  $\mathcal{U}$ .

Type formers:

- Π-types
- Σ-types
- identity types
- quotients

Types:

- the empty type
- the unit type
- the booleans
- the natural numbers

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## Computational behavior of El

The decoding of the types and type formers is definitional.

For  $x : \mathcal{V}^0$ ,  $y : \text{El } x \to \mathcal{V}^0$ :  $\text{El}(\Pi^0 \times y) \equiv \prod_{a:\text{El } x} \text{El}(y a)$  The hierarchy of iterative sets,  $\mathcal{V}^0$ 

V<sup>0</sup> as a Tarski universe

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# $\mathcal{V}^0$ is a universe of h-sets

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The decoding El x of any  $x : \mathcal{V}^0$  is an h-set.



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#### Proof.

By construction, El x embeds into  $\mathcal{V}^0$ , which is an h-set.

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### Univalence

#### Univalence(?)

Is the universe  $\mathcal{V}^0$  univalent? No. For  $x, y : \mathcal{V}^0$ , the type x = y is an h-proposition, while El  $x \simeq$  El y is in general a proper h-set.



The hierarchy of iterative sets,  $\mathcal{V}^0$ 

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# $\overline{\mathsf{P}\mathsf{rec}}\mathsf{ategory}\ \mathsf{structure}\ \mathsf{on}\ \mathcal{V}^0$

### Definition (Precategory structure on $\mathcal{V}^0$ )

The precategory  $\mathbf{V}^0$  is given by

- $Ob(\mathbf{V}^0) := \mathcal{V}^0$
- $\operatorname{Hom}_{\mathbf{V}^0}(x, y) := \operatorname{El} x \to \operatorname{El} y$

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#### Theorem

 $\mathbf{V}^0$  is finitely cocomplete and locally cartesian closed.

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# CwF structure

We can *almost* define a CwF structure on **HSet**:

- Ty  $\Gamma := \Gamma \rightarrow \mathcal{HSet}$
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 $\mathcal{V}^0$  is an h-set, so we can construct the corresponding CwF structure:

- Ty  $\Gamma := \mathsf{EI} \ \Gamma \to \mathcal{V}^0$
- $\operatorname{Tm}(\Gamma, A) := \prod_{x: \mathsf{El} \Gamma} \mathsf{El}(Ax)$

The rest of the construction is analogous to the one for **HSet**.



V<sup>0</sup> as a Tarski universe

 $\mathcal{V}^0$  as a precategory and its Rezk completion  $\mathcal{O} \cap \mathcal{O}$ 

# Relationship between **V**<sup>0</sup> and **HSet**

El extends to a functor  $V^0 \to HSet,$  which is full and faithful. Is HSet the Rezk completion of  $V^0?$ 



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Theorem (de Jong-Kraus-Forsberg-Xu)

The HoTT book V is equivalent to the type of covered marked extensional well-founded orders.

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The assumption that El is surjective corresponds to Shulman's axiom of well-founded materialisation.

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# Summary

 $\bullet \ \mathcal{V}^0$  as a Tarski universe and as a precategory

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- $\bullet \ \mathcal{V}^0$  as a Tarski universe and as a precategory
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  - Higher h-level generalisation:  $\mathcal{V}^n$