The category of iterative sets in HoTT

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Outline

The hierarchy of iterative sets

Recall the hierarchy of iterative sets:

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Definition (The hierarchy of iterative sets)

We define the following predicate on Aczel's type $W_{A:\mathcal{U}} A$:

$$\begin{aligned}
&\text{iterative-set} : W_{A:\mathcal{U}} A \to \text{Type} \\
&\text{iterative-set}\ (\text{sup}\ A\ f) := \text{is-embedding}\ f \times \prod_{a:A} \text{iterative-set}\ (f\ a)
\end{aligned}$$

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and take the corresponding subtype of $W_{A:\mathcal{U}}A$:

$$\mathcal{V}^0 := \sum_{x : W_{A:\mathcal{U}}A} \text{iterative-set } x$$

$\mathcal{V}^0$ is an h-set

Theorem

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Proof.

Sketch:

- $\mathcal{V}^0$ is a fixpoint of the functor $X \mapsto \sum_{A:U} (A \hookrightarrow X)$
- For $(A, f), (B, g) : \sum_{A:U} \left(A \hookrightarrow \mathcal{V}^0\right)$ we have:

$$((A, f) = (B, g)) \simeq \prod_{x:\mathcal{V}^0} (\text{fiber } f\ x \simeq \text{fiber } g\ x)$$

- All the fibers are propositions

□

$\mathcal{V}^0$ as a Tarski universe

A Tarski universe is a type with a type family.

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Definition

We define the following decoding function:

$$\begin{aligned} &\mathsf{El} : \mathcal{V}^0 \to \mathcal{U} \\ &\mathsf{El}\ (\mathsf{sup}\ A\ f,\ p) := A \end{aligned}$$

Types and type formers in $\mathcal{V}^0$

We have the following in $\mathcal{V}^0$ if we have them in $\mathcal{U}$.

Type formers:

- Π-types
- Σ-types
- identity types
- quotients

Types:

- the empty type
- the unit type
- the booleans
- the natural numbers

Computational behavior of El

The decoding of the types and type formers is definitional.

For $x : \mathcal{V}^0$, $y : \mathsf{El}\ x \to \mathcal{V}^0$:

$$\mathsf{El}\,(\Pi^0\ x\ y) \equiv \prod_{a : \mathsf{El}\ x} \mathsf{El}\,(y\ a)$$

$\mathcal{V}^0$ is a universe of h-sets

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The decoding $\mathrm{El}\ x$ *of any* $x : \mathcal{V}^0$ *is an h-set.*

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Proof.

By construction, El x embeds into $\mathcal{V}^0$, which is an h-set. □

Univalence

Univalence(?)

Is the universe $\mathcal{V}^0$ univalent? No. For $x, y : \mathcal{V}^0$, the type $x = y$ is an h-proposition, while $\mathrm{El}\ x \simeq \mathrm{El}\ y$ is in general a proper h-set.

Precategory structure on $\mathcal{V}^0$

Definition (Precategory structure on $\mathcal{V}^0$)

The precategory $\mathbf{V}^0$ is given by

- $\mathrm{Ob}(\mathbf{V}^0) :\equiv \mathcal{V}^0$
- $\mathrm{Hom}_{\mathbf{V}^0}(x, y) :\equiv \mathrm{El}\ x \to \mathrm{El}\ y$

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Theorem

$\mathbf{V}^0$ is finitely cocomplete and locally cartesian closed.

CwF structure

We can *almost* define a CwF structure on **HSet**:

- $\mathsf{Ty}\ \Gamma := \Gamma \to \mathcal{HS}et$
- $\mathsf{Tm}\,(\Gamma, A) := \prod_{x:\Gamma} A\, x$

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$\mathcal{V}^0$ is an h-set, so we can construct the corresponding CwF structure:

- $\mathsf{Ty}\,\Gamma := \mathsf{El}\,\Gamma \to \mathcal{V}^0$
- $\mathsf{Tm}\,(\Gamma, A) := \prod_{x:\mathsf{El}\,\Gamma} \mathsf{El}\,(A\,x)$

The rest of the construction is analogous to the one for **HSet**.

Relationship between $\mathbf{V}^0$ and **HSet**

El extends to a functor $\mathbf{V}^0 \to \mathbf{HSet}$, which is full and faithful. Is **HSet** the Rezk completion of $\mathbf{V}^0$?

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The assumption that El is surjective corresponds to Shulman's axiom of well-founded materialisation.

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  - CwF structure on presheaves on $\mathbf{V}^0$
  - Higher h-level generalisation: $\mathcal{V}^n$