On epimorphisms and acyclic types in univalent mathematics

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We study epimorphisms and acyclic types in univalent mathematics. The epimorphisms are of course a natural object of study, the definition being that a map $f : A \to B$ is an *epimorphism* if for every type X, the precomposition map

$$(B \to X) \xrightarrow{(-)\circ f} (A \to X) \tag{1}$$

is an embedding. In other words, extensions of maps out of A through B are unique when they exist.

The epimorphisms behave quite differently in higher types than do the surjections of sets, with the difference manifesting itself starting at the level of 2-types. To bring out this structure, we introduce the notion of k-*epimorphism*, which is a map f such that (1) is an embedding for all k-types X.

We prove that being a (k-)epimorphism is a fiberwise condition. The 0-epimorphisms are the surjections, i.e., the (-1)-connected maps, and the 1-epimorphisms are the 0-connected maps. But the 2-epimorphisms are the maps whose fibers are connected and whose fundamental groups are *perfect*. (A group G is perfect if its abelianization is trivial.)

This connects the notion of epimorphism with that of *acyclic* types. These are classically the types A whose reduced integral homology vanishes, $\tilde{\mathrm{H}}_n(A) \simeq 0$ for all $n \geq 0$. Likewise, the acyclic maps, which are related to Quillen's *plus construction* and the Kan–Thurston theorem [KT76], are characterized as those inducing isomorphisms on homology groups for all local abelian coefficient systems. These characterizations only work assuming Whitehead's principle, and the correct definition in univalent mathematics is that A is acyclic if its suspension, ΣA, is contractible. We prove:

Proposition 1. *A map $A \to \mathbf{1}$ is k-epic if and only if ΣA is k-connected, and an epimorphism if and only ΣA is contractible.*

(Recall that a type is k-connected if and only if its k-truncation is contractible.) Correspondingly, we call a type A k-*acyclic* if ΣA is k-connected.

A group G is determined by its classifying type $\mathrm{B}G$, which is a pointed, connected 1-type with fundamental group G. We know that the abelianization G^{ab} has classifying type $\mathrm{B}^2 G^{\mathrm{ab}} \simeq \|\Sigma \mathrm{B}G\|_2$ [BvDR18], and thus we obtain:

Corollary 2. *For a group G, the map $\mathrm{B}G \to \mathbf{1}$ is 2-epic if and only if G is perfect.*

Next we begin the search for examples of acyclic types. These cannot be sets because of the following:

Theorem 3. *For a set A, the following are equivalent:*
(i) A is 1-acyclic;
(ii) A is acyclic;
(iii) A is contractible.

Interestingly, to prove this, we need the fact that the generators map $\eta : A \to \mathrm{F}_A$, from A to the free group on A is an injection [MRR88, Chapter X]. This has previously been proved in homotopy type theory [BCDE21].

We have the following simple inclusions, all of which are strict in general:

$$\begin{array}{ccc} \{\, k\text{-connected maps} \,\} & \hookrightarrow & \{\, (k+1)\text{-epis} \,\} \\ \downarrow & & \downarrow \\ \{\, k\text{-equivalences} \,\} & \hookrightarrow & \{\, k\text{-epis} \,\} \end{array}$$

We also note the following, whose proof is short enough to fit here:

Lemma 4. *Any acyclic and simply connected type A is infinitely connected.*

Proof. This follows by the Freudenthal suspension theorem: The unit map of the loop-suspension adjunction, $\sigma : A \to \Omega\Sigma A$, is $2n$-connected whenever A is n-connected (for $n \geq 0$). By acyclicity, $\Sigma A = \mathbf{1}$, so $\Omega\Sigma A = \mathbf{1}$, so the connectivity of σ is that of A. Starting with the assumption that A is 1-connected, we in turn conclude that A is 2-, then 4-, etc., hence 2^n-connected, for all n. ☐

Classically, it's well known that the Higman group [Hig51], given by the presentation

$$H = \langle\, a, b, c, d \mid a = [d, a], b = [a, b], c = [b, c], d = [c, d] \,\rangle$$

is acyclic (i.e., has an acyclic classifying space), and moreover, this presentation is *aspheric*, meaning that the presentation complex is already a 1-type [DV73]. The presentation complex is easily imported into homotopy type theory as the higher inductive type $\mathrm{B}H$ with a point constructor $\mathrm{pt} : \mathrm{B}H$, four path constructors $a, b, c, d : \Omega\mathrm{B}H$, and four 2-cell constructors corresponding to the relations. We prove:

Proposition 5. *The type $\mathrm{B}H$ is acyclic and $\pi_1(\mathrm{B}H)$ is the Higman group.*

We leave for future work the matter of developing enough combinatorial group theory to show that $\mathrm{B}H$ is a 1-type.

We note that nullification at $\mathrm{B}H$ provides a nontrivial modality L, whose subuniverse of L-separated types consists of all types, following [CR22, Ex. 6.6].

Future directions Classically, the acyclic maps form the left class of an accessible orthogonal factorization system, whose right class are the *hypoabelian* maps (i.e., whose π_1-kernels have no non-trivial perfect subgroups). This factorization system exists in all $(\infty, 1)$-toposes [Hoy19], and can be used to derive the McDuff–Segal completion theorem, but with arguments that don't lend themselves to direct internalization in homotopy type theory.

We only prove the main closure properties of the acyclic maps as one should expect from such a left class. We hope in future work to construct the factorization system using Quillen's plus construction. To do so, we expect to need the strengthening of Lemma 4 stating that acyclic, simply connected types are actually contractible. It appears to be an open problem whether this holds in all $(\infty, 1)$-toposes. It follows from Whitehead's principle, but also holds in $(\infty, 1)$-toposes where this fails, such as that of parametrized spectra. So we propose to assume it as a principle, which we might call the *plus principle*.

We would also like to investigate whether other classically known acyclic types and maps can be shown to be so in homotopy type theory, such as $\mathrm{B}\,\mathrm{Aut}(\mathbb{N})$ [dlHM83] and $\mathrm{B}\Sigma_\infty \to (\Omega^\infty\Sigma^\infty\mathbb{S}^0)_0$, the Barratt–Priddy(–Quillen) theorem [BP72].

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