Smash Products Are Symmetric Monoidal in HoTT

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In his 2016 proof of $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$, Brunerie [Bru16] crucially uses—but never proves—that the smash product is symmetric monoidal. Due to the vast amount of path algebra involved when reasoning about smash products in HoTT, this has since remained open. While it turns out that smash products are not needed to complete Brunerie's proof [LM23; BLM22], the problem is still interesting in its own right. Several attempts have been made at salvaging the situation. Floris Van Doorn came very close to a complete proof by considering an argument using closed monoidal categories but left a gap where the path algebra got too technical. Another line of attack by Cavallo and Harper [CH20; Cav21a] is the addition of parametricity which can be used to provide a solution in an extension of HoTT (but not in plain HoTT). In this talk, we introduce a heuristic for reasoning about functions defined over smash products and use it to give a complete proof of the fact that the smash product is symmetric monoidal. The main result has been formalised in `Cubical Agda`[1], but here we introduce the proof in plain book HoTT as it does not rely crucially on any cubical features.

The model of the smash product we will use here is given by the cofibre of the inclusion $A \vee B \to A \times B$. For the sake of clarity, let us spell this out in detail:

Definition 1. *The smash product of two pointed types A and B is the HIT generated by*

- *a point $\star_\wedge : A \wedge B$*
- *for every pair $(a, b) : A \times B$, a point $\langle a, b\rangle : A \wedge B$*
- *for every point $a : A$, a path $\mathsf{push}_\mathsf{l}(a) : \langle a, \star_B\rangle = \star_\wedge$*
- *for every point $b : B$, a path $\mathsf{push}_\mathsf{r}(b) : \langle \star_A, b\rangle = \star_\wedge$*
- *a coherence $\mathsf{push}_\mathsf{lr} : \mathsf{push}_\mathsf{l}(\star_A) = \mathsf{push}_\mathsf{r}(\star_B)$.*

The fact that the smash product is commutative is very direct. Its associativity, however, is harder to prove. The first proof in HoTT was given by van Doorn [Doo18], using the adjunction $(A \wedge B \to_\star C) \simeq (A \to_\star (B \to_\star C))$. Later, Brunerie [Bru18] provided another solution, using a computer generated proof in `Agda`. Here, we give an explicit proof by considering a more involved HIT, denoted $\bigwedge(A, B, C)$, satisfying $A \wedge (B \wedge C) \simeq \bigwedge(A, B, C)$ and (by design) $\bigwedge(A, B, C) \simeq \bigwedge(C, A, B)$. This automatically gives the desired equivalence $\alpha_{A,B,C} : A \wedge (B \wedge C) \simeq (A \wedge B) \wedge C$. The advantage of an explicitly described equivalence is that it becomes easier to trace. In particular, it is easy to understand its behaviour on point constructors and 1-dimensional paths, which will turn out to be precisely what we need.

The key problem in proving the fact that the smash product is symmetric monoidal is verifying Mac Lane's pengaton[2], i.e. the (pointed) commutativity of the following diagram.

$$
\begin{tikzcd}
A \wedge (B \wedge (C \wedge D)) \arrow[rr, "\mathsf{id}_A \wedge \alpha_{B,C,D}"] \arrow[d, "\alpha_{A,B,C\wedge D}"'] & & A \wedge ((B \wedge C) \wedge D) \arrow[d, "\alpha_{A,B\wedge C,D}"] \\
(A \wedge B) \wedge (C \wedge D) \arrow[dr, "\alpha_{A\wedge B,C,D}"'] & & (A \wedge (B \wedge C)) \wedge D \arrow[dl, "\alpha_{A,B,C} \wedge \mathsf{id}_D"] \\
& ((A \wedge B) \wedge C) \wedge D &
\end{tikzcd}
$$

A naive proof attempt by $\wedge$-induction forces us to fill (highly non-trivial) cubes up to 5 dimensions and we quickly descend into coherence hell. As a first attempt at remedying this, we introduce the following useful lemma.

[1] The formalisation is available at https://github.com/aljungstrom/cubical/blob/pentagon/Cubical/HITs/SmashProduct/SymmetricMonoidal.agda (L562). Note that the definition of *precategories* is non-standard.

[2] To the reader upset with the somewhat non-standard direction of the arrows, we apologise sincerely. It is done this way to better match the formalisation.

Lemma 1. *Let $A \tilde{\wedge} B$ denote the cofibre of the inclusion $A + B \to A \times B$ (this is precisely $A \wedge B$ without the $\mathsf{push}_{\mathsf{lr}}$ constructor). The inclusion $\iota : A \tilde{\wedge} B \to A \wedge B$ equalises any $f, g : A \wedge B \to C$.*

Lemma 1 is still not quite strong enough for our purposes, so let us try to rework it. Recall that a pointed type A is called *homogeneous* if for any point $a : A$ we have an equality of pointed types $(A, \star_A) = (A, a)$. We turn our attention to an incredibly useful lemma, first conjectured for Eilenberg-MacLane spaces in work leading up to [BLM22] and soon thereafter proved and generalised by Cavallo [Cav21b] (and later further generalised by Buchholtz, Christensen, G. Taxerås Flaten and Rijke [Buc+23]), which states that any two pointed functions $f, g : A \to_\star B$ with B homogeneous are equal as pointed functions iff their underlying functions are equal. Using the fact that $A \to_\star B$ is homogeneous whenever B is homogeneous, the adjunction $(A \wedge B \to_\star C) \simeq (A \to_\star (B \to_\star C))$ yields the following result.

Lemma 2. *Let $f, g : A \wedge B \to_\star C$ with C homogeneous. We have $f = g$ as pointed functions iff $f\langle a, b\rangle = g\langle a, b\rangle$ for all $(a, b) : A \times B$.*

If we could apply Lemma 2 to functions $A \wedge (B \wedge (C \wedge D)) \to ((A \wedge B) \wedge C) \wedge D$, the pentagon would be trivial. Unfortunately, arbitrary smash products are *not* necessarily homogeneous. Fortunately, there is still some use for the lemma. Let us consider the following construction.

Definition 2. *Let $f, g : A \wedge B \to C$ and let $h : (a, b) : A \times B \to f\langle a, b\rangle = g\langle a, b\rangle$. We define the two maps $L_h : A \to_\star f(\star_\wedge) = g(\star_\wedge)$ and $R_h : B \to_\star f(\star_\wedge) = g(\star_\wedge)$ by*

$$L_h(a) = \mathsf{ap}_f(\mathsf{push}_{\mathsf{l}}(a))^{-1} \cdot h(a, \star_B) \cdot \mathsf{ap}_g(\mathsf{push}_{\mathsf{l}}(a))$$
$$R_h(b) = \mathsf{ap}_f(\mathsf{push}_{\mathsf{r}}(b))^{-1} \cdot h(\star_A, b) \cdot \mathsf{ap}_g(\mathsf{push}_{\mathsf{r}}(b))$$

where we may simply take $f(\star_\wedge) = g(\star_\wedge)$ to be pointed by either $L_h(\star_A)$ or $R_h(\star_B)$ (these are equal by $\mathsf{push}_{\mathsf{lr}}$, so the choice does not matter).

The following result is a weakening of Lemma 1.

Lemma 3. *Let $f, g : A \wedge B \to C$. The following data gives an equality $f = g$:*

- *A homotopy $h : ((a, b) : A \times B) \to f\langle a, b\rangle = g\langle a, b\rangle$*
- *Equalities of pointed functions $L_h = \mathsf{const}_{L_h(\star_B)}$ and $R_h = \mathsf{const}_{R_r(\star_B)}$.*

The second datum above looks almost absurd. It is asking us to provide equalities of pointed functions which is much stronger than what is actually needed. We need not worry: the codomain of these functions is homogeneous, so it suffices to provide unpointed equalities. However, there is a point to the pointedness requirement: when e.g. B is another smash product, as is the case in the pentagon, Lemma 2 applies. This makes this part of the proof very direct. In fact, Lemma 3 and Lemma 2 may be iteratively applied to any n-fold smash product (let us take smash products to be right-associative) without ever increasing the dimension of the proof goal. Let us state this as an informal theorem:

Theorem 1 (Informal)**.** *To show that two functions $f, g : A_1 \wedge (A_2 \wedge (\cdots \wedge A_n)) \to B$ are equal, it suffices to provide a family of paths $f\langle x_1, \ldots, x_n\rangle = g\langle x_1, \ldots, x_n\rangle$ for $x_i : A_i$ and to show that it is coherent with f and g on any* single *application of $\mathsf{push}_{\mathsf{l}}$ or $\mathsf{push}_{\mathsf{r}}$ (e.g. $\mathsf{ap}_{\langle x_1, \ldots, x_{i-1}, -\rangle}(\mathsf{push}_{\mathsf{r}}\langle x_{i+1}, \ldots, x_n\rangle)$).*

This theorem/heuristic gives a very direct proof of the pentagon. The pentagon holds by definition for homogeneous elements $\langle a, b, c, d\rangle : A \wedge (B \wedge (C \wedge D))$, so we are only left to trace single instances of the push constructors. This turns out to be straightforward (albeit somewhat lengthy) due to the low dimension of the proof goals. The pointedness requirement holds almost by definition. Theorem 1 can also be used to show the remaining axioms, including the hexagon identity, and we arrive at the main result:

Theorem 2. *The smash product is monoidal symmetric with the type of booleans as unit.*

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