The Category Interpretation of (Polarized and Directed) Type Theory

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Workshop on Homotopy Type Theory Univalent Foundations

Motivation For the entirety of its (young) life, homotopy type theory has always been closely connected to the theory of groupoids. The first glimpse of HoTT's central themes came in Hofmann and Streicher's pioneering work[HS95], which explicitly used groupoids as a metatheory for Martin-Löf Type Theory with higher types. As HoTT emerged, it was quickly noted [vdBG10, AR12] that the identity types in HoTT function as a formalism for synthetic higher groupoids. This viewpoint was subsequently incorporated into the HoTT Book as the section "Types are higher groupoids"[Uni13, Sect. 2.1].

Groupoids, of course, are just categories with an added 'symmetry' assumption: that all morphisms are invertible. If we drop this assumption, we obtain ordinary category theory. In recent years, there have been a number of investigations[LH11, Nuy15, Nor19, RS17, WL20] into the possibility of **directed homotopy type theory**, that is, homotopy type theory where we drop the assumption that identity types are symmetric. Such type theories promise to have the same relationship to (higher) category theory that ordinary, *undirected* HoTT has to higher groupoid theory.

In this talk, we hope to conduct the directed analogue of foundational work by Dybjer, Hofmann, Streicher, and others [Dyb95, Hof97, HS95] which prepared the way for undirected HoTT. In particular, we investigate the **category model**, a directed analogue of Hofmann and Streicher's groupoid model, which serves to motivate a syntax for a directed MLTT. This model also serves as the paradigm example of two notions, **polarized categories with families (PCwFs)** and **directed categories with families (DCwFs)**, which form the core of our model theory. The establishment of this foundation paves the way for a number of exciting developments, including the use of *directed path induction* to prove naturality "for free" in the development of synthetic category theory[Alt19], the development of polarized and directed higher-order abstract syntax, and the possibility of directed higher observational type theory,[1] à la Altenkirch, Kaposi, and Shulman[Shu22, AKS22].

A Polarized Lambda Calculus To be able to conduct metatheoretic investigations into type theory, it is necessary to have an adequate notion of a *model* of type theory. One of the most prominent such notions in the literature are **categories with families (CwFs)**, originally introduced by Dybjer[Dyb95]. Indeed, in order to establish the independence of Streicher's Axiom K [Str93] from the usual rules of MLTT, Hofmann and Streicher [HS95] utilize the formalism of CwFs: they define a *countermodel* CwF which validates the rules of MLTT but refutes K. This countermodel – the groupoid model – interprets contexts as groupoids, substitutions as functors between groupoids, types in context Δ as Δ-indexed families of groupoids, and so on. This will be our starting-off point.

We can obtain a new CwF, which we'll call the category model, simply by replacing every instance of "groupoid" with "category" in the definition of the groupoid model.[2] However, this structure is more than just a CwF: there are a number of operations we can define on this model which didn't exist (or, rather, were trivial) in the groupoid case. First among these is the operation of taking *opposite categories*: we obtain 'negation' operations on contexts and types, as well as a

[1] As speculated on in the previous HoTT/UF Workshop [Neu22]

[2] This CwF does not necessarily support the same type formers as the groupoid model, specifically Π-types and identity types, but this is what we'll work to address.

negative context extension operation

$$\frac{\Delta \colon \mathsf{Con}}{\Delta^- \colon \mathsf{Con}} \qquad \frac{A \colon \mathsf{Ty}\ \Delta}{A^- \colon \mathsf{Ty}\ \Delta} \qquad \frac{\Delta \colon \mathsf{Con} \quad A \colon \mathsf{Ty}\ \Delta^-}{\Delta \triangleright^- A \colon \mathsf{Con}}$$

subject to various governing equations obtained from careful study of the category model. We show that such constructions are necessary in order to define function spaces on the category model: the polarity induced by functions (i.e. positive and negative occurrences) is suppressed by the symmetry present in the groupoid model, but here must be dealt with explicitly. Abstracting the key features of how these negation operations works, we obtain our first semantic notion: polarized CwFs. Polarized CwFs are the appropriate model for type theories equipped with a 'polarity calculus' of positive and negative contexts and types. Polarized CwFs will be the basic foundation upon which we build the more sophisticated semantic notions necessary for directed type theory.

Cores and Directed Type Theory Another aspect of the category model which we'll want to abstractly characterize is the fact that the groupoid model 'lives inside' the category model. As a first pass, we can note that the underlying category of contexts in the groupoid model (i.e. Grpd) is a coreflective subcategory[3] of the category model's category of contexts (i.e. Cat) – the coreflector is the *core groupoid* operation, which sends a category to its maximal subgroupoid. This fact can be represented as extending the syntax of polarized type theory with a 'core' operation on contexts, plus some extra structure. But the coreflective subcategory relationship plays out on the type- and term-level as well. In particular, we get **core types**, whose terms can be used both positively and negatively

$$\frac{A \colon \mathsf{Ty}\ \Delta}{A^0 \colon \mathsf{Ty}\ \Delta} \qquad \frac{a \colon \mathsf{Tm}\ \Delta\ A^0}{+a \colon \mathsf{Tm}\ \Delta\ A \qquad -a \colon \mathsf{Tm}\ \Delta\ A^-}$$

Finally, once we have established the rules for polarized dependent type theory with cores, we are able to give rules for the directed analogue of identity types, **hom types**, which supply a framework for synthetic category theory. The formation and introduction rules are as follows.

$$\frac{A \colon \mathsf{Ty}\ \Delta \quad a_0 \colon A^- \quad a_1 \colon A}{(a_0 \Rightarrow_A a_1) \colon \mathsf{Ty}\ \Delta} \qquad \frac{a \colon \mathsf{Tm}\ \Delta\ A^0}{\mathsf{refl}_a \colon \mathsf{Tm}\ \Delta\ (-a \Rightarrow_A +a)}$$

In the first rule, note the essential use of polarized types to indicate that a_0 occurs negatively and a_1 occurs positively. In the second, note the necessity of core types, since the term a needs to appear both positively and negatively. All these constructs together constitutes the definition of a *directed CwF*, the appropriate notion for semantic investigations into directed type theory.

Time allowing, we discuss the J-rule for hom types, a directed form of path induction, and how it can be used to prove the naturality of transformations between functors in our synthetic category theory.

Future Research Since this work is intended to lay a foundation, there are naturally several further avenues of research to explore. These include:

- Making sense of universes in directed type theory (e.g. devising a directed analogue to the universe construction of [HS99]), stating directed univalence, and comparing this formulation of directed univalence to existing statements (e.g. of [WL20])
- Investigating further the theory of higher hom types and directed HITs, beginning with a proof that the category model refutes the directed analogue of Axiom K. Analogously to the

[3] Also a reflective subcategory, though we don't explore this aspect.

use of ordinary HoTT to study higher groups [BvDR18] and the fundamental group(s) of a space[Uni13, Sect. 6.11], directed HoTT could presumably furnish a theory of higher monoids and the 'fundamental monoid' of a directed space.

- Developing a polarized higher-order abstract syntax (in the style of [BKS21, BKS23]) with semantics in presheaf categories[Hof99],[4] and leveraging it to conduct a directed analogue to higher observational type theory [Shu22, AKS22].

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