NATURAL NUMBERS FROM INTEGERS

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Background. When presenting type theory, one typically includes basic type formers like Π, Σ, and $=$, as well as standard finite types $\mathbf{0}$, $\mathbf{1}$, and $\mathbf{2}$. These type formers can all be interpreted in finite sets, and so do not suffice to define any infinite types. Hence one separately postulates inductive types, allowing one to define for example the type of natural numbers, which is the canonical example of an infinite type.

In a homotopical setting the situation changes. Using higher inductive types, we can define the circle S^1 as being freely generated by a point $\mathsf{pt} : S^1$ and an identification $\mathsf{loop} : \mathsf{pt} = \mathsf{pt}$. This definition is finitary in the sense that it does not use any recursive constructors; non-recursive inductive types can all be interpreted in finite sets. While the circle itself is finite in the sense that it has only one path component, one expects its loop space $\mathsf{pt} = \mathsf{pt}$ to be the integers, an infinite type. More precisely, we say an integer type is any type $\mathbb{Z}$ freely generated by an element $0 : \mathbb{Z}$ and an automorphism $S : \mathbb{Z} \simeq \mathbb{Z}$. This notion can be compared with that of a natural number type, which is any type $\mathbb{N}$ freely generated by an element $0 : \mathbb{N}$ and an endomorphism $S : \mathbb{N} \to \mathbb{N}$. Given an appropriate large elimination principle for the circle, one can show that its loop space $\mathsf{pt} = \mathsf{pt}$ is an integer type [1]. Thus the circle seems to be a source of infinity.

Now one can ask if it is possible to construct a natural number type from an integer type. This question was raised by Egbert Rijke and Mike Shulman on the nForum [3], who gave a solution assuming an impredicative type of propositions. Impredicativity is a source of inductive definitions, so it makes the problem significantly easier. Robert Rose went much further in his PhD thesis [4]. Suprising many experts, he gave a predicative construction of the naturals from the integers. However, Rose's construction is quite complicated and needs two univalent universes (only the outer one may be replaced by large elimination).

Our contribution is to provide a new, much simpler construction of naturals from integers. Our construction is also more general: we do not need any universes. Instead, we rely on coproducts with large elimination. Our construction does not use any homotopical principles and is valid also in extensional type theory. Notably, the principles we use to construct naturals from integers are weaker than those used to construct the circle and to show that its loop space is an integer type.

Our result has the following categorical analogue. Suppose we have a locally cartesian closed $(\infty, 1)$-category with finite coproducts that satisfy descent (type-theoretically, descent corresponds to large elimination). Then given an integer object, we have a natural number object. In particular, any locally cartesian closed $(\infty, 1)$-category with finite colimits that satisfy descent has a natural number object. This is a strengthening of the result in [2], which assumes impredicativity in the form of a subobject classifier.

Constructing the naturals. Suppose $\mathbb{Z}$ is an integer type; that is, $\mathbb{Z}$ is a type freely generated by an element $0 : \mathbb{Z}$ and an automorphism $S : \mathbb{Z} \simeq \mathbb{Z}$. Our goal is to construct a natural number type. Note that it is not clear how to prove directly basic properties about $\mathbb{Z}$, such as the fact that $\mathbb{Z}$ is a set or that it has decidable equality. For the naturals, this is easy to prove by induction. But the induction principle for $\mathbb{Z}$ does not allow a direct proof.

To motivate our construction, it is helpful to think about how to understand the integers given a natural number type $\mathbb{N}$. In this case, we have an equivalence $\mathbf{1} + \mathbb{N} \simeq \mathbb{N}$, which can be used to define an automorphism $\mathbb{N} + \mathbf{1} + \mathbb{N} \simeq \mathbb{N} + \mathbf{1} + \mathbb{N}$. This induces a map $\mathbb{Z} \to \mathbb{N} + \mathbf{1} + \mathbb{N}$, corresponding to the fact that any integer is either $-n-1$ for a natural n, 0, or $n+1$ for a natural n. Crucially for us, we can perform the same construction given any types A, B with $A+B \simeq B$ and an element $a : A$. While it is not clear how to show $\mathbf{1} + \mathbb{Z} \simeq \mathbb{Z}$, we do have the following.

Lemma 1. *There is an equivalence $\mathbb{Z} + \mathbb{Z} \simeq \mathbb{Z}$.*

Proof sketch. The idea is to decompose the integers into odd and even integers, which can be done directly by integer induction. □

From this we obtain an automorphism $\mathbb{Z} + \mathbb{Z} + \mathbb{Z} \simeq \mathbb{Z} + \mathbb{Z} + \mathbb{Z}$, sending the first summand into the first two and the last two summands into the third. Since we have also a point of the middle summand $\mathbb{Z}$, we obtain a map $\mathbb{Z} \to \mathbb{Z} + \mathbb{Z} + \mathbb{Z}$, and hence a map $\mathbb{Z} \to \mathbf{1} + \mathbf{1} + \mathbf{1}$. This induces a decomposition $\mathbb{Z} \simeq \mathbb{Z}^- + \mathbb{Z}^0 + \mathbb{Z}^+$ of the integers into three parts (this step requires large elimination for coproducts), such that 0 lies in $\mathbb{Z}^0$, and S restricts to an equivalence $\mathbb{Z}^0 + \mathbb{Z}^+ \simeq \mathbb{Z}^+$. Note that the map $\mathbb{Z} \to \mathbf{1} + \mathbf{1} + \mathbf{1}$ giving the sign of an integer cannot be defined directly by integer recursion; we needed the intermediate steps.

Let M denote the coproduct $\mathbb{Z}^0 + \mathbb{Z}^+$. By construction, 0 and S restrict to $0 : M$ and $S : M \to M$. While one does expect M to be a natural number type, it is not clear how to show this directly due to some coherence issues. Namely, it is not clear that $\mathbb{Z}^0$ is contractible. Instead, we construct the natural numbers as a Σ-type over M.

Lemma 2. *On M, we have a decidable binary relation $\leq$ with the following properties: if $x \leq y$ then $x \leq S(y)$; if $S(x) \leq y$ then $x \leq y$; $x \leq y$ iff $S(x) \leq S(y)$; $x \leq 0$ iff $x \in \mathbb{Z}^0$; and finally $x \leq x$ for all x.*

Note that we do not claim $\leq$ is transitive. Indeed this is not direct.

Proof. First define integer subtraction, by recursion. Then take $x \leq y$ to mean $x - y \in \mathbb{Z}^- + \mathbb{Z}^0$. □

Given this ordering on M, we can talk about initial segments of M and functions defined on them. More precisely, suppose $A(m)$ is a type family over $m : M$, together with functions $0_A : \Pi_{x:\mathbb{Z}^0} A(x)$ and $S_A : \Pi_{x:M} A(x) \to A(S(x))$. Note that 0_A is more than just an element of $A(0)$. Given $u : M$, we say a section $f : \Pi_{x:M}(x \leq u) \to A(x)$ of A defined on the initial segment up to u is *inductive* when we have paths $f(x, -) = 0_A(x)$ for $x : \mathbb{Z}^0$, $x \leq u$ and $f(S(x), -) = S_A(f(x, -))$ for $S(x) \leq u$. We write $I(u)$ for the type of inductive functions defined up to u.

In particular, we can take $A(m) \equiv_{\mathsf{def}} M$, $0_A(-) \equiv_{\mathsf{def}} 0$, and $S_A(-) \equiv_{\mathsf{def}} S$. We say $m : M$ is a *standard natural* when we have an inductive function f defined up to m and a path $f(m, -) = m$. We write $N(m)$ for the type of proofs that m is a standard natural. Finally we arrive at our definition of the type $\mathbb{N}$: it is $\Sigma_{m:M} N(m)$. In order to show that $\mathbb{N}$ is a natural number type, the following lemma is key.

Lemma 3. *For A, 0_A, S_A as above, we have that $I(0)$ is contractible, and $I(u) \simeq I(S(u))$.*

The proof of this lemma is surprisingly simple with the following characterisation of inductive functions. A section of A defined up to $S(u)$ can be *restricted* to a section defined up to u, since $x \leq S(u)$ implies $x \leq u$. Conversely, a section defined up to u can be *extended* to a section defined up to $S(u)$, using 0_A and S_A. Then a function is inductive when it equals the restriction of its extension. With this characterisation, $I(0)$ is contractible since it is the type of fixed points of a constant map. We have $I(u) \simeq I(S(u))$ since restriction commutes with extension in the appropriate sence, and $s \circ r$ and $r \circ s$ have equivalent types of fixed points whenever both composites make sense.

We use the lemma twice: to define $0_N : N(0)$ and $S_N : N(m) \simeq N(S(m))$, and then again to prove the desired induction principle for $\mathbb{N}$. In order to define a section of a type family over $\mathbb{N}$, we claim that up to any $u : M$, there is a unique inductive section of a certain A, 0_A, S_A. By phrasing the goal in this more structured way, the inductive step becomes reversible, since $I(u) \simeq I(S(u))$. This allows us to reduce natural number induction to integer induction.

References

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