Rigidification of cartesian closed ∞-functors

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The so-called internal language conjecture (see [KL18]) relating models of some type theories to some classes of $(\infty, 1)$-categories includes the following statement:

Conjecture 1
The functor Ho_∞ restricts to DK-equivalences

$$\iota : \mathbf{CompCat}_{\Sigma,Id} \to \mathbf{Qcat}_{lex}$$
$$\iota_\pi : \mathbf{CompCat}_{\Sigma,\Pi_{ext},Id} \to \mathbf{Qcat}_{lcc}$$

where the domain of ι is the relative category whose objects are comprehension categories modeling Σ and identity types, and the domain of ι_π is the relative category of comprehension categories additionally modeling Π-types.

The statement involving the first functor has been proven in [KS19], while the second functor has been shown to indeed factor through the relative category $\mathbf{Qcat}_{lcc}$ (taking values in locally cartesian closed quasicategories) in [Kap15].

In [Che22], we showed that this second functor happened to be essentially surjective on objects, and mistakenly claimed that the second point of the conjecture followed directly. Indeed, the suggested argument was not complete as a morphism rigidification method was missing to obtain the expected result, which is the purpose of the present work. Precisely, since $\mathbf{CompCat}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}}$ is a non-full subcategory of $\mathbf{CompCat}_{\Sigma,\mathrm{Id}}$, and since, similarly, $\mathbf{Qcat}_{lcc}$ is a non-full subcategory of $\mathbf{Qcat}_{lex}$, it was unclear what the connection was between the hom-spaces $Hom_\pi(X,Y)$ (where the subscript refers to preservation of Π-types/dependent products) computed in any of these two relative subcategories and the hom-spaces $Hom(X,Y)$ computed in the whole relative categories.

However, if we defined a subcategory

$$\mathbf{CompCat}^{w}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}} \hookrightarrow \mathbf{CompCat}_{\Sigma,\mathrm{Id}}$$

with the same object as $\mathbf{CompCat}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}}$ but with the morphism only required to preserve Π-types *up to equivalence* (in the sense that such a morphism $F : \mathcal{M} \to \mathcal{M}'$ is required to induced a cartesian closed ∞-functor $Ho_\infty F$), a hom-space $Hom(X,Y)$ in $\mathbf{CompCat}^{w}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}}$ would be a subspace of the hom-space computed in $\mathbf{CompCat}_{\Sigma,\mathrm{Id}}$, given as the union of a collection of connected components (that is because if F is such that $Ho_\infty F$ is cartesian closed, and F' is connected to F as an object of the hom-space $Hom(X,Y)$, $Ho_\infty F'$ will necessarily be cartesian closed).

With such a definition, from the first functor being a DK-equivalence one could deduce that the weak equivalence between spaces

$$Hom(\mathcal{M}, \mathcal{M}') \simeq Hom(Ho_\infty\mathcal{M}, Ho_\infty\mathcal{M}')$$

restricts to the relevant connected components as to provide a weak equivalence

$$Hom_\pi(\mathcal{M}, \mathcal{M}') \simeq Hom_\pi(Ho_\infty\mathcal{M}, Ho_\infty\mathcal{M}')$$

Since ι_π is known to be essentially surjective on objects, this would give a DK-equivalence

$$\iota'_\pi : \mathbf{CompCat}^{w}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}} \to \mathbf{Qcat}_{lcc}$$

Therefore, the second point of the conjecture reduces to prove that the non-full inclusion

$$\mathbf{CompCat}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}} \hookrightarrow \mathbf{CompCat}^{w}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}}$$

is itself a DK-equivalence, which in turn boils to the following functor also being a DK-equivalence:

$$\mathbf{Trb}_\pi \to \mathbf{Trb}^{\mathbf{w}}_\pi$$

where $\mathbf{Trb}_\pi$ is the usual category of π-tribes and cartesian closed morphisms of tribes between them (as introduced in [Joy17]), and $\mathbf{Trb}^{\mathbf{w}}_\pi$ is a weakened version where the morphism are only supposed to induce cartesian closed ∞-functor between the underlying $(\infty,1)$-categories of the domain/codomain π-tribes.

The main difficulty to tackle this question lies in the fact that these categories equipped with the suitable notion of weak equivalence do not seem to provide a notion of homotopy structure easy to work with (concretely, those are not fibration categories).

While there is direct workaround to induce a fibration category structure on the category **FibCat** of fibration categories (as shown in [Szu16]), one may also have to consider a DK-equivalent category which can be endowed with such a fibration category structure. This is done in [KS19] by replacing the category of tribes **Trb** by the category **sTrb** of semi-simplicial tribes.

Here we proceed similarly by rather working with two DK-equivalent fibration categories $\mathbf{hTrb}_\pi$ and $\mathbf{hTrb}^{\mathbf{w}}_\pi$ that are full subcategories of the previous ones.

The crucial (but simple) technical observation which allows us to prove these two categories to be DK-equivalent is then the following:

Lemma 2
Suppose $f : \mathcal{T} \to \mathcal{S}$ is a morphism between π-tribes such that $\mathbf{Ho}_\infty(f)$ is a cartesian closed ∞-functor and consider the pullback square below where $P\mathcal{S}$ is the π-tribe whose objects are span of trivial fibrations in $\mathcal{S}$

$$\begin{tikzcd}
\mathcal{T}' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P\mathcal{S} \arrow[d] \\
\mathcal{T} \times \mathcal{S} \arrow[r, "f \times id_{\mathcal{S}}"'] & \mathcal{S} \times \mathcal{S}
\end{tikzcd}$$

Then $\mathcal{T}'$ is a π-tribe equivalent to $\mathcal{T}$ and the morphisms $\mathcal{T}' \to \mathcal{T}$ and $\mathcal{T}' \to \mathcal{S}$ are π-closed.

We also believe this technique can be used for other features of type theory (e.g. natural numbers).

References

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