Higher Coherence Equations of Semi-Simplicial Types as n-Cubes of Proofs*

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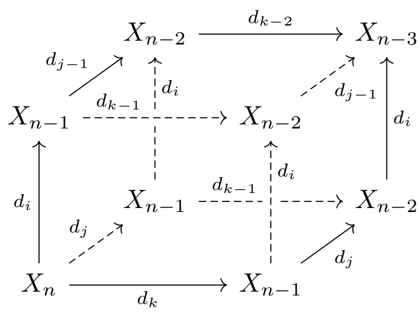
1 Introduction

The construction of *semi-simplicial types* in type theory as dependent families of types [8] turned out to be remarkably difficult in spite of considerable scrutiny over the past decade (see [7] for an overview). In a proof-relevant setting, the seemingly innocent *semi-simplicial identity* would be witnessed by a family of terms $\alpha_{i,j} : d_i d_j =_{X_n \to X_{n-2}} d_{j-1} d_i$ (for any $0 \leq i < j \leq n$) which may combine together and create new proof terms.



Cube of proofs. Let us illustrate this with the figure opposite. Each face of the cube is associated with a proof term α. There are exactly two ways to deform the lower "meridian" $d_i d_j d_k$ of the cube into the upper one $d_{k-2} d_{j-1} d_i$ relative to X_n, X_{n-3}, either by passing through the backmost faces or through the frontmost ones—both "hemispheres" of the cube. Now, these two ways correspond exactly to the two proofs of the equality $d_i d_j d_k =_{X_n \to X_{n-3}} d_{k-2} d_{j-1} d_i$, given by the compositions $\pi := \alpha_{j,k} * \alpha_{i,k-1} * \alpha_{i,j}$ and $\pi' := \alpha_{i,j} * \alpha_{i,k} * \alpha_{j-1,k-1}$ (up to whiskering). For the equality to be *coherent* at this stage would require to "fill the cube", i.e. exhibit a term $\beta_{i,j,k} : \pi = \pi'$ witnessing the fact that both proofs are indeed equal. Of course the β's would, in turn, fit onto the "3-hemispheres" of a 4-cube and require a term e.g. $\gamma_{i,j,k,\ell}$ to identify their possible compositions—then the process repeats with 5-cubes, 6-cubes, and so on *ad infinitum*.

Related work. One solution to handle these infinite towers of coherence consists in adding some form of strictness to equality. The first co-author considered, for example, the case of dependent families of *sets* (i.e. 0-truncated types)—see [4], or [5] for a joint work with Ramachandra. Altenkirch, Capriotti and Kraus, on the other hand, extended HoTT with a second equality type which is strict [2]. A whole different approach recently presented at HoTTEST by Kolomatskaia extends type theory with a construction to inhabit dependent streams of types [6].

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Our approach. We adopt the "n-cubes of proofs" point of view as above. Proof terms fit inside of an infinite collection of n-cubes, and how they may combine together is a consequence of the combinatorial structure of n-cubes. Roughly speaking, an n-cube would be the data of a term $c_n : h^+_{n,n-1} =_{\dots} h^-_{n,n-1}$ identifying both of its $(n-1)$-hemispheres, which are given by composing the $(n-1)$-faces together in some way. The terms $h^+_{n,n-1}, h^-_{n,n-1}$ are proofs of equality themselves, identifying both of the $(n-2)$-hemispheres $h^+_{n,n-2}, h^-_{n,n-2}$. The general picture would be a tower of equalities:

$$c_n : h^+_{n,n-1} =_{\left(h^+_{n,n-2} =_{\left(h^+_{n,n-3} =_{\dots} h^-_{n,n-3}\right)} h^-_{n,n-2}\right)} h^-_{n,n-1}$$

We suspect that examining the exact combinatorial structure[1] of n-cubes and their k-hemispheres will bring us one step closer to understanding all the higher coherence equations necessary to fully construct semi-simplicial types *without the need of univalence*, i.e. internal to MLTT extended with a notion of "infinite streams of data".

2 Hemispheres of the n-Cube

Given some way to encode the faces of an n-cube, a k-hemisphere (relative to the n-cube) is a specific collection of k-faces that "fit nicely together". A first step thus consists in determining what these k-faces might be.

Definition. *We denote by $(D^n_k, \leq)$ the poset of* increasing *sequences $x_1 \dots x_n$ of length n with values $0 \leq x_i \leq k$. The order on D^n_k is defined* pointwise, *i.e. $x \leq y \iff \forall\, i,\, x_i \leq y_i$.*

The D^n_k's are connected by two notable order-preserving maps: there is a canonical inclusion of D^n_k into D^n_{k+1} which we will write d_*, as well as a map $R : D^n_k \to D^{n+1}_k$ which adds the value k at the end of a sequence. Moreover, we have the following inductive description:

Lemma. $D^n_k = d_* D^n_{k-1} \amalg R D^{n-1}_k$

Our main result so far is the following:

Theorem. *The k-hemispheres of the n-cube are exactly described by D^k_{n-k}. Moreover, all the possible ways to compose the k-faces are given by all the* topological sorts *on D^k_{n-k}.*

We have a working algorithm that generates all the k-faces of both k-hemispheres. It is additionally possible to choose a total order on D^k_{n-k} which is compatible with $\leq$ and gives a canonical choice of topological sort. Several questions that are open:

1. Compute the appropriate whiskering required for composition of k-faces to make sense in type theory.

2. How to deal with the possible choices of composition? Can we show them to be equivalent without the need of new proof terms (e.g. the Eckmann-Hilton argument)?

3. Using the new insights brought by the "n-cubes of proofs" point of view, is it now possible to construct semi-simplicial types in a proof assistant?

We are hopefully close to giving a simple procedure that addresses the first question. This is an ongoing work which may be subject to new developments by the end of April.

[1] Discussions with Métayer revealed that some of this combinatorial structure have already been spelled out over the study of *orientals* (e.g. the work of Aitchison [1]). See [9] for the original article on orientals, and [3] for a recent work of Ara, Lafont and Métayer.

References

[1] Iain R. Aitchison. The geometry of oriented cubes, 2010. Original 1986 report with cosmetic improvements. Available at: https://arxiv.org/abs/1008.1714.

[2] Thorsten Altenkirch, Paolo Capriotti, and Nicolai Kraus. Extending Homotopy Type Theory with Strict Equality. In *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*, volume 62 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 21:1–21:17. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016. doi:10.4230/LIPIcs.CSL.2016.21.

[3] Dimitri Ara, Yves Lafont, and François Métayer. Orientals as free algebras. Preprint, 2022. Available at: http://arxiv.org/abs/2209.08022.

[4] Hugo Herbelin. A dependently-typed construction of semi-simplicial types. *Mathematical Structures in Computer Science*, 25(5):1116–1131, 2015. doi:10.1017/S0960129514000528.

[5] Hugo Herbelin and Ramkumar Ramachandra. A parametricity-based formalization of semi-simplicial and semi-cubical sets. Preprint, 2023. Available at: https://pauillac.inria.fr/~herbelin/articles/nu-types-draft23.pdf.

[6] Astra Kolomatskaia. Semi-Simplicial Types. HoTTEST, December 2022. Available at: https://github.com/FrozenWinters/SSTs.

[7] Nicolai Kraus. On the Role of Semisimplicial Types. Abstract, TYPES 2018. Available at: https://www.cs.nott.ac.uk/~psznk/docs/on_semisimplicial_types.pdf.

[8] Peter LeFanu Lumsdaine. Semi-simplicial types. Online, 2013. Available at: https://ncatlab.org/ufias2012/published/Semi-simplicial+types.

[9] Ross Street. The algebra of oriented simplexes. *Journal of Pure and Applied Algebra*, 49(3):283–335, 1987. doi:10.1016/0022-4049(87)90137-x.