Classification of Covering Spaces and Canonical Change of Basepoint

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Homotopy Type Theory is used as a synthetic language for homotopy theory [Uni13, LB13, HFFLL16]. Its synthetic nature allows one to manipulate the nuts and bolts of homotopy directly. As such, the mathematical arguments keep a strong connection to the underlying geometric ideas.

With the desire to gain better insight into how to approach homotopy theory in HoTT, we set out to prove that there is no degree one map from a closed oriented genus g surface to a closed oriented genus h surface if $g < h$. Although we have not reached that destination, along the way we have proved synthetic versions of some classical results.

Building on Hou (Favonia) and Harper's results on covering spaces [HH18], we prove HoTT versions of the lifting criterion and the classification of covering spaces; although these are already shown in HoTT [BvDR18, Thrm. 7], the proofs we provide are more basic and might be more accessible. Secondly, we show when there exists canonical change-of-basepoint isomorphisms $\pi_n(X, a) \cong \pi_n(X, b)$. Large parts of the theory are formalized in Coq, see https://gitlab.tue.nl/computer-verified-proofs/covering-spaces.

Classification of Covering Spaces. Before one can prove results from classical homotopy theory in HoTT, one needs to find proper translations of these results. One might try a direct translation first: the classical results can be stated in HoTT using existing definitions of the fundamental group and induced maps from the HoTT book [Uni13]. The new language also allows one to express the underlying ideas in different ways.

Using Hou (Favonia) and Harper's definition of pointed covering spaces in HoTT [HH18, Def. 7], we prove a direct translation of the lifting criterion.

Lemma 1 (Lifting Criterion, cf. [Hat01, Prop. 1.33])**.** *Let $F : X \to \mathsf{Set}$ with $u_0 : F(x_0)$ be a pointed covering space over a pointed type (X, x_0). A pointed map $f : (Y, y_0) \cdot\!\!\to (X, x_0)$, with Y a connected type, can be lifted to a map $\tilde{f} : (Y, y_0) \cdot\!\!\to (\sum_X F, (x_0, u_0))$ if and only if*

$$f_*(\pi_1(Y, y_0)) \subset \mathsf{pr_1}_*(\pi_1(\textstyle\sum_X F, (x_0, u_0))) \ . \tag{1}$$

Here f_* and $\mathsf{pr_1}_*$ denote the induced maps on the fundamental groups, not the shorthand notation for transport as is common in HoTT.

In proving this statement, we found criterion (1) inconvenient to work with in HoTT. The notation $f_*(\pi_1(Y, y_0))$, for example, conceals multiple truncations — a propositional-truncation to define the image of a map and a set-truncation for π_1 — which hinder access to the homotopical objects. We therefore proved that criterion (1) is equivalent to another condition, one tailored to HoTT.

Lemma 2. *Let $F : X \to \mathsf{Set}$ with $u_0 : F(x_0)$ be a pointed covering space over a pointed type (X, x_0) and let $f : (Y, y_0) \cdot\!\!\to (X, x_0)$ be a pointed map. Then the criterion $f_*(\pi_1(Y, y_0)) \subset \mathsf{pr_1}_*(\sum_X F, (x_0, u_0))$ is equivalent to the condition that for all loops $p : y_0 =_Y y_0$ there exists a loop from u_0 to u_0 lying over $f_*(p)$ in F, meaning that*

$$\mathsf{transport}^F(f_*(p), u_0) =_{F(x_0)} u_0 \ .$$

Together with the universal covering space constructed by Hou and Harper [HH18, Thrm. 13], we use the alternative lifting criterion (Lemma 2) to show that connected, pointed covering spaces are classified by subgroups of $\pi_1(X, x_0)$.

Theorem 3 (Classification, cf. [Hat01, first half of Thm. 1.38])**.** *Let (X, x_0) be a connected, pointed type. Then there is an equivalence between pointed, connected covering spaces (F, u_0) over (X, x_0) and subgroups of $\pi_1(X, x_0)$, obtained by associating to the covering space (F, u_0) the subgroup given by the predicate*

$$|p|_0 \mapsto \left(\mathsf{transport}^F(p, u_0) =_{F(x_0)} u_0\right) \ ,$$

meaning that $|p|_0 : \pi_1(X, x_0)$ belongs to the subgroup if there exists a loop from u_0 to u_0 lying over p.

Canonical Change of Basepoint. In classical homotopy theory, a path p from a to b in a topological space X induces a change-of-basepoint isomorphism between homotopy groups $\pi_n(X, a) \cong \pi_n(X, b)$. The isomorphism depends on the homotopy class of the path p. In the case that X is simply-connected, the isomorphism can be considered canonical — there is only one homotopy class of paths from a to b.

In HoTT, transport along a path $p : a =_X b$ also gives rise to an isomorphism $\pi_n(X, a) \cong \pi_n(X, b)$. Often we do not have access to an explicit path $p : a =_X b$, but only know the truncation $\|a =_X b\|$ to be inhabited. In these cases, we can use *extension by weak constancy* [HH18, Lemma 6]: there exists a canonical isomorphism $\pi_n(X, a) \cong \pi_n(X, b)$ if transport along all paths $p, q : a =_X b$ yields the same results, i.e.

$$\mathsf{transport}^{\pi_n(X,-)}(p, -) \ = \ \mathsf{transport}^{\pi_n(X,-)}(q, -) \ .$$

This is equivalent to stating that the fundamental group $\pi_1(X, a)$ acts trivially on the higher homotopy groups $\pi_n(X, a)$.

We prove the following statements for when the π_1-action is trivial, and hence for when there exists canonical change-of-basepoint isomorphisms.

Theorem 4. *Let X be a type with designated point $a : X$.*

(i) If X is simply-connected, then the action of $\pi_1(X, a)$ on $\pi_n(X, a)$ is trivial for all $n \geq 1$;

(ii) The fundamental group $\pi_1(X, a)$ is abelian if and only if the action on itself is trivial;

(iii) If the type $\prod_{p,q : \Omega(X,a)} p \cdot q = q \cdot p$ is merely inhabited, then the action of $\pi_1(X, a)$ on $\pi_n(X, a)$ is trivial for all $n \geq 1$.

Results (i) and (ii) are easily shown, both in the classical theory and in HoTT. For result (iii), we use the relationship between transport in consecutive loop spaces given below; it follows from [Uni13, Thrm. 2.11.4]. Loop spaces that satisfy the assumption in (iii) are classically called *homotopy* commutative as the commutativity may only hold up to homotopy; this is the default setting in HoTT.

Lemma 5. *Let $p : a =_X b$ and $u : \Omega^{n+1}(X, a)$, then we have the following equality in $\Omega(X, b)$:*

$$\mathsf{transport}^{\Omega^{n+1}(X,-)}(p, u) = (\mathsf{apd}_{\mathsf{refl}^n_{(-)}}(p))^{-1} \cdot \mathsf{ap}_{\mathsf{transport}^{\Omega^n(X,-)}(p,-)}(u) \cdot \mathsf{apd}_{\mathsf{refl}^n_{(-)}}(p) \ .$$

The term $\mathsf{apd}_{\mathsf{refl}^n_{(-)}}(p)$ can be thought of as a homotopy between the transported n-cell $p_*(\mathsf{refl}^n_a)$ at b and the constant n-cell refl^n_b at b.

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