Set-theoretic pluralism and homotopy type theory

Matteo de Ceglie

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In recent years, the debate around the set-theoretic foundations of mathematics has been focussed on the idea of *pluralism*, first introduced by Hamkins (2012). According to pluralism, the universe of set theory is not unique and determined, but it is a collection of several, different universes, all equally legitimate. This is opposed to the standard position known as *universism*, according to which there exists only one, unique set-theoretic universe, namely the cumulative hierarchy V. The fact that ZFC doesn't fully determine V is not a problem: sooner or later, we will find the right extension of ZFC such that we can settle all the open questions and determine V. On the other hand, pluralists argue against such a position, pointing to the numerous mutually incompatible models of ZFC (and its extensions) that have been developed and to the current practice of forcing. Moreover, they proposed several mathematically characterisations of these philosophical ideas: the set-theoretic multiverses.[1] In the last decade, several have been proposed and developed, and this novel research field has brought several interesting insights in how set theory works, for example with the results in set-theoretic geology and the modal logic of forcing.[2]

This debate is still exclusively present in set theory. There are a couple of contributions regarding pluralism and non-classical set theories[3], but almost nothing regarding pluralism and homotopy type theory. The only research area that tries to consider pluralism together with homotopy type theory consists in arguments aimed at introducing the possibility of a pluralist foundation of mathematics that encompasses both set theoretic foundations and univalent foundations (this is, for example, one of the goals of Friend (2014)). Another cluster of contributions focusses on comparing the foundational strength of set theory against the one of univalent foundations.[4]

In this paper, my goal is to introduce set-theoretic pluralism in the homotopy type theory conception of set. I argue that this move can give us important insights of the nature of set-theoretic pluralism. Before proceeding, an important disclaimer is warranted. In this paper I am *not* trying to define a plural collection of models of homotopy type theory using forcing, in an analogous manner as forcing is used in set theory to produce different non-standard models. Using Kripke-Joyal forcing it is possible to do so (see Awodey, Gambino, and Hazratpour (2021)), and this could open up the possibility of introducing independent

[1] See for example Gitman and Hamkins (2011), Steel (2014), Antos et al. (2018), and Väänänen (2014).
[2] See for example Fuchs, Hamkins, and Reitz (2015) and Hamkins and Linnebo (2022).
[3] See Tarafder and Venturi (2021) and Jockwich, Tarafder, and Venturi (2022).
[4] See for example Maddy (2017).

proofs in homotopy type theory (in a context similar to Tarafder and Venturi (2021), that introduced independence proofs in constructive set theory). This is surely a very interesting research question, but not the focus of this paper. To reiterate, the aim of this paper is to introduce the idea of set-theoretic pluralism in the conception of set as characterised in homotopy type theory.

To do so, consider the definition of cumulative hierarchy in homotopy type theory. According to this definition, a cumulative hierarchy V is a higher inductive type inside some universe $\mathcal{U}$, defined with 3 constructors. These 3 constructors essentially give us the possibility of building a type *set* from a type $A : \mathcal{U}$ and a function f, that two of these sets are equal when their images under f are equal, and finally a 0-truncation constructor sich that V is a h-set (a type whose all identity types are mere propositions).[5] In this paper I investigate the possibility of defining different cumulative hierarchies. To do so, we need to add some more constructors to the definition, in such a way that we end up with slightly different cumulative hierarchies. However, this must be done carefully: we need the new set of constructors to still define a cumulative hierarchy and to be not trivial (i.e. if the constructors contradicts the ambient homotopy set theory). To exemplify this point, I will consider the case of the Continuum Hypothesis, and how we can convert it in two mutually incompatible constructors to define two different cumulative hierarchies.

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[5]For the details, see for example Ledent (2014).

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