Enriched Categories in Univalent Foundations

Niels van der Weide

Radboud Universiteit, Nijmegen, The Netherlands
nweide@cs.ru.nl

Enriched categories have found numerous applications, including effects in programming languages [EMS14, PP01], abstract homotopy theory [GJ09], and in higher category theory [Lur09]. In this abstract, we discuss an ongoing formalization of enriched categories in univalent foundations. More specifically, we define the notion of univalence for enriched categories, and we prove that the bicategory of univalent enriched category is univalent. This gives us a structure identity principle for enriched categories. The definitions and theorems in this abstract are formalized in the Coq proof assistant [Tea22] using the UniMath library [VAG+] and building upon [AKS15, WMA22].

1 Univalent Enriched Categories

In the remainder of this abstract, we fix a monoidal category $\mathcal{V}$, and we denote its unit by $\mathbb{1}$ and the tensor by $\otimes$. Usually, the definition of an enriched category is a slight modification of the notion of category: the homs are required to be objects of $\mathcal{V}$ instead of sets. However, we take a different approach, which is based on the notion of *enrichment*. Since every enriched category C has an underlying category C_0, there is a 2-functor $(-)_0$ from the 2-category $\mathcal{V}\mathsf{Cat}$ of categories enriched over $\mathcal{V}$ to the 2-category Cat of categories [Kel82]. The idea is that an enrichment of C is an object of the fiber of $(-)_0$ along C.

Definition 1. A **$\mathcal{V}$-enrichment** E of a category C consists of

- a function $\mathsf{E}(-,-) : \mathsf{C} \to \mathsf{C} \to \mathcal{V}$;
- for all $x : \mathsf{C}$ a morphism $\mathsf{Id} : \mathbb{1} \to \mathsf{E}(x,x)$ in $\mathcal{V}$;
- for all $x, y, z : \mathsf{C}$ a morphism $\mathsf{Comp} : \mathsf{E}(y,z) \otimes \mathsf{E}(x,y) \to \mathsf{E}(y,z)$ in $\mathcal{V}$;
- functions $\mathsf{FromArr} : \mathsf{C}(x,y) \to \mathcal{V}(\mathbb{1}, \mathsf{E}(x,y))$ and $\mathsf{ToArr} : \mathcal{V}(\mathbb{1}, \mathsf{E}(x,y)) \to \mathsf{C}(x,y)$ for all $x, y : \mathsf{C}$

such that the usual axioms for enriched categories are satisfied and such that $\mathsf{FromArr}$ and ToArr are inverses of each other. A **category with a $\mathcal{V}$-enrichment** is a pair of a category C together with a $\mathcal{V}$-enrichment of C.

Note that enrichments of categories have been considered in other work as well [MU22], although they used yet another definition. The reason why we choose to define enriched categories this way, is because using enrichments, we can define a displayed bicategory $\mathcal{V}\mathsf{UnivCat}_{\mathsf{disp}}$ over $\mathsf{UnivCat}$ whose total bicategory is the bicategory $\mathcal{V}\mathsf{UnivCat}$ of enriched categories (Definition 4). This way the proof of the univalence for the bicategory of enriched categories becomes simpler, because we can reuse the proof that the bicategory of categories is univalent [AFM+21]. Note that our notion of categories with a $\mathcal{V}$-enrichment is actually equivalent to the usual notion of enriched categories.

Proposition 2. *The type of categories with a $\mathcal{V}$-enrichment is equivalent to the type of $\mathcal{V}$-enriched categories defined using the definition given by Kelly [Kel82].*

Next we define *univalent enriched categories*. With our definition of enrichments, we say that a univalent enriched category is a univalent category together with an enrichment. Equivalently, we also phrase univalence for enriched categories as defined in [Kel82]: such an enriched category C would be univalent if the underlying category C_0 is univalent.

Definition 3. A **univalent $\mathcal{V}$-enriched category** is a pair of a univalent category C together with a $\mathcal{V}$-enrichment of C.

2 The Bicategory of Univalent Enriched Categories

Next we construct the bicategory of univalent enriched categories, and we prove that this bicategory is univalent. To define this bicategory, we use displayed bicategories [AFM+21], and thus we need to define enrichments for functors and natural transformations. Concretely, we need to define $\mathcal{V}$-enrichments for functors $\mathsf{F} : \mathsf{C}_1 \rightarrow \mathsf{C}_2$ from E_1 to E_2, where E_1 and E_2 are $\mathcal{V}$-enrichments of C_1 and C_2 respectively. We also need to define $\mathcal{V}$-naturality for natural transformations. The definitions of these notions are in a similar style as Definition 1, and for the precise definitions, we refer the reader to the formalization.

Definition 4. We define the displayed bicategory $\mathcal{V}\mathsf{UnivCat}_{\mathsf{disp}}$ over $\mathsf{UnivCat}$ as follows:

- The displayed objects over a category C are $\mathcal{V}$-enrichments of C;
- The displayed 1-cells over a functor $\mathsf{F} : \mathsf{C}_1 \rightarrow \mathsf{C}_2$ from E_1 to E_2 are $\mathcal{V}$-enrichments of F;
- The displayed 2-cells over a natural transformation τ are proofs that τ is a $\mathcal{V}$-natural.

The total bicategory of $\mathcal{V}\mathsf{UnivCat}_{\mathsf{disp}}$ is the bicategory of univalent enriched categories, and we denote it by $\mathcal{V}\mathsf{UnivCat}$.

The proof that the data in Definition 4 actually forms a displayed bicategory, is similar to the construction of the bicategory of enriched categories in set-theoretic foundations. We conclude this abstract by proving that $\mathcal{V}\mathsf{UnivCat}$ is univalent.

Lemma 5. *If $\mathcal{V}$ is univalent, then the displayed bicategory $\mathcal{V}\mathsf{UnivCat}_{\mathsf{disp}}$ is univalent.*

Theorem 6. *If $\mathcal{V}$ is univalent, then the bicategory $\mathcal{V}\mathsf{UnivCat}$ is univalent.*

The methods used to prove Theorem 6 are similar to the methods used for proofs of univalence in [AFM+21]. Concretely, this theorem says that two enriched categories are equal if we have an enriched equivalence between them. As such, we obtain a structure identity principle for enriched categories.

There are numerous way to extend the work in this abstract. One particular way, is by instantiating the formal theory of monads to enriched categories [Str72, vdW22]. Concretely, this means that one constructs the enriched Eilenberg-Moore and Kleisli category for an enriched monad. The usual theorems about monads and adjunctions for enriched categories would then follow from the formal theory developed by Street [Str72], and these theorems are useful for formalizing results about the semantics of the extended effect calculus [EMS14].

References

[AFM+21] Benedikt Ahrens, Dan Frumin, Marco Maggesi, Niccolò Veltri, and Niels van der Weide. Bicategories in univalent foundations. *Math. Struct. Comput. Sci.*, 31(10):1232–1269, 2021.

[AKS15] Benedikt Ahrens, Krzysztof Kapulkin, and Michael Shulman. Univalent categories and the rezk completion. *Mathematical Structures in Computer Science*, 25(5):1010–1039, 2015.

[EMS14] Jeff Egger, Rasmus Ejlers Møgelberg, and Alex Simpson. The enriched effect calculus: syntax and semantics. *J. Log. Comput.*, 24(3):615–654, 2014.

[GJ09] Paul G Goerss and John F Jardine. *Simplicial homotopy theory*. Springer Science & Business Media, 2009.

[Kel82] Max Kelly. *Basic concepts of enriched category theory*, volume 64. CUP Archive, 1982.

[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.

[MU22] Dylan McDermott and Tarmo Uustalu. What makes a strong monad? In Jeremy Gibbons and Max S. New, editors, *Proceedings Ninth Workshop on Mathematically Structured Functional Programming, MSFP@ETAPS 2022, Munich, Germany, 2nd April 2022*, volume 360 of *EPTCS*, pages 113–133, 2022.

[PP01] Gordon D. Plotkin and John Power. Adequacy for algebraic effects. In Furio Honsell and Marino Miculan, editors, *Foundations of Software Science and Computation Structures, 4th International Conference, FOSSACS 2001 Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2001 Genova, Italy, April 2-6, 2001, Proceedings*, volume 2030 of *Lecture Notes in Computer Science*, pages 1–24. Springer, 2001.

[Str72] Ross Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2(2):149–168, 1972.

[Tea22] The Coq Development Team. The Coq Proof Assistant, January 2022.

[VAG+] Vladimir Voevodsky, Benedikt Ahrens, Daniel Grayson, et al. UniMath — a computer-checked library of univalent mathematics. available at http://unimath.org.

[vdW22] Niels van der Weide. The Formal Theory of Monads, Univalently. *CoRR*, abs/2212.08515, 2022.

[WMA22] Kobe Wullaert, Ralph Matthes, and Benedikt Ahrens. Univalent monoidal categories. *arXiv preprint arXiv:2212.03146*, 2022.