Rezk Completion of Bicategories

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In this work in progress, we aim to show that every not necessarily univalent bicategory admits a Rezk completion. To do so, we show how every bicategory is weakly bi-equivalent to a univalent one. This result has been formalized in `UniMath`. It remains to show that this univalent bicategory satisfies a suitable universal property. Furthermore, we characterize when the Rezk completion of a functor category is the functor category of the Rezk completions. Consequently, we conclude that the Rezk completion of the bicategory of not-necessarily univalent categories is not bi-equivalent to the bicategory of univalent categories.

1 Rezk Completion of Bicategories

In [1], it is shown that every locally univalent bicategory is weakly bi-equivalent to a univalent bicategory. However, not every bicategory is locally univalent. Therefore, it remained an open problem whether or not every bicategory is weakly bi-equivalent to a locally univalent bicategory. In [3], it is shown how every monoidal category is weakly equivalent to a univalent monoidal category. Considering monoidal categories as one-object bicategories, the work presented there shows how every one-object bicategory is weakly bi-equivalent to a (locally) univalent one-object bicategory. In this work, we have shown that the approach used in [3] generalizes to (arbitrary) bicategories in a straightforward way.

A locally univalent bicategory is a bicategory such that every hom-category is univalent. Therefore, we expect that it suffices to construct the *local* Rezk completion as the bicategory with the same objects, but where each hom-category is replaced by the Rezk completion of that hom-category:

Theorem 1. *(LRB,LRB_is_locally_univalent, psfunctor_B_to_LRB_is_weak_biequivalence) Let $\mathcal{B}$ be a bicategory. The following data can be given the structure of a locally univalent bicategory $\hat{\mathcal{B}}$:*

$$\mathsf{ob}(\hat{\mathcal{B}}) \equiv \mathsf{ob}(\mathcal{B}), \quad \hat{\mathcal{B}}(x,y) \equiv \mathsf{RC}(\mathcal{B}(x,y)),$$

where RC *is the Rezk completion of categories [2]. Furthermore, the identity on objects and the weak equivalence given by the Rezk completion of each hom-category induces a weak bi-equivalence $\mathcal{B} \to \hat{\mathcal{B}}$.*

The composition of $\hat{\mathcal{B}}$ is defined using the universal property of the Rezk completion, analogously to how the tensor of the monoidal Rezk completion was defined.

By first applying, to any bicategory, the construction in Theorem 1 and then applying the construction in construction 5.13 of [1], we conclude:

Theorem 2. *(rezk_completion_2) For every bicategory $\mathcal{B}$, there is a univalent bicategory $\mathsf{RC}_2(\mathcal{B})$ together with a weak bi-equivalence $\eta_{\mathcal{B}} : \mathcal{B} \to \mathsf{RC}_2(\mathcal{B})$.*

2 Universal property of the Rezk completion

The universal property of the Rezk completion of categories is:

Theorem 3 ([2, Thm. 8.4])**.** *Let $\mathcal{C}$ be a category and $\eta_{\mathcal{C}} : \mathcal{C} \to \mathsf{RC}(\mathcal{C})$ be the Rezk completion of $\mathcal{C}$. Every functor $F : \mathcal{C} \to \mathcal{D}$ into a univalent category $\mathcal{D}$, factorizes uniquely through $\eta_{\mathcal{C}}$. Equivalently, pre-composing with $\eta_{\mathcal{C}}$ is an isomorphism of categories:*

$$\eta_{\mathcal{C}} \cdot - : [\mathsf{RC}(\mathcal{C}), \mathcal{D}] \to [\mathcal{C}, \mathcal{D}].$$

We expect that the Rezk completion of bicategories satisfies an analogous universal property:

Conjecture 1. Let $\mathcal{B}$ be a bicategory and $\eta_{\mathcal{B}} : \mathcal{B} \to \mathsf{RC}_2(\mathcal{B})$ be the Rezk completion of $\mathcal{B}$. For any univalent bicategory $\mathcal{D}$, the pseudo-functor

$$\eta_{\mathcal{B}} \cdot - : [\mathsf{RC}_2(\mathcal{B}), \mathcal{D}] \to [\mathcal{B}, \mathcal{D}]$$

is a bi-equivalence of bicategories. Furthermore, this characterizes the $(\mathsf{RC}_2(\mathcal{B}), \eta_{\mathcal{B}})$ uniquely.

Theorem 3 means precisely that the inclusion of the full sub-bicategory $\mathsf{Cat}_{\mathsf{univ}}$ of univalent categories of the bicategory Cat has a left bi-adjoint. Analogously, we suspect that Conjecture 1 can also be characterized in the language of tricategories. That is, that we get a tri-adjunction between the tricategory of bicategories and the tricategory of univalent bicategories.

3 Towards computing (local) Rezk completions

Now that we know that any bicategory is equivalent to a (locally and globally) univalent bicategory, we would like to compute those Rezk completions. Moreover, one would expect to be able to first compute the local Rezk completion and than compute the global Rezk completion of that. In order to compute the local Rezk completion, one needs to compute the Rezk completion of hom-categories. We will see that even in the case of Cat, this is not as well-behaved as one might hope. Usually, if a category $\mathcal{D}$ satisfies some property, then so does $[\mathcal{C}, \mathcal{D}]$ (for any category $\mathcal{C}$). This is not the case for the Rezk completions. In order to conclude this, we have the following lemma:

Lemma 1. Let $\mathcal{B}$ be a full sub-bicategory of Cat. The following are equivalent:

1. $\prod_{\mathcal{C},\mathcal{D}:\mathcal{B}} \mathsf{RC}([\mathcal{C}, \mathcal{D}]) = [\mathcal{C}, \mathsf{RC}(\mathcal{D})]$
2. $\mathsf{RC}(\mathcal{B}) = \mathcal{B}_{\mathsf{univ}}$ (where $\mathcal{B}_{\mathsf{univ}}$ is the intersection of $\mathcal{B}$ and $\mathsf{Cat}_{\mathsf{univ}}$)
3. $\mathcal{B} \simeq \mathcal{B}_{\mathsf{univ}}$

Because Cat is (trivially) a full sub-bicategory of Cat, we conclude that functor categories are not (always) computed by taking the Rezk completion on the codomain. Indeed, not every category is (strongly) equivalent to a univalent one. Consequently, we also do not have that the unique functor from Cat to $\mathsf{Cat}_{\mathsf{univ}}$ is a weak equivalence.

By defining $\mathcal{B}$ to be a full sub-bicategory consisting of two objects, we conclude from Lemma 1:

Proposition 1. For every $\mathcal{C}$ and $\mathcal{D}$, the following are equivalent:

1. $\mathsf{RC}([\mathcal{C}, \mathcal{D}]) = [\mathcal{C}, \mathsf{RC}(\mathcal{D})]$
2. $\mathcal{C}$ and $\mathcal{D}$ are each equivalent to a univalent category

References

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