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Colimits in the category of pointed types

Homotopy type theory is a useful system for developing synthetic homotopy theory. We treat types as spaces and function types as hom-groupoids, thereby making the universe of types an internal version of the ∞-category of spaces. In certain cases, we want to prove things about the category of *pointed* types and basepoint-preserving functions $A \to_* B :\equiv \sum_{f:A\to B} f(a_0) = b_0$. In our ongoing work on a type-theoretic version of the Brown representability theorem, we need a Yoneda-like lemma whose standard proof requires pointed function types to take colimits to limits. Since we must carry proofs of the identity $f(a_0) = b_0$, this property is tricky to prove compared to the unpointed version. To prove it, we must place coherence conditions on the colimits we take.

We exhibit a large class of colimits preserved by pointed function types. These colimits are taken over graphs equipped with *strong-tree* structure, called *strong trees* for short. Such colimits include, for example, pushouts, sequential colimits, and wedge sums. Our main theorem, then, is constructing colimits of diagrams F over strong trees in the pointed category as colimits of F in the unpointed category, where the latter can be formed as ordinary pushouts.

All strong trees are trees in the traditional sense, i.e., have contractible quotients. It is unclear, however, whether the converse holds. Our development relies on the combinatorial flavor of strong-tree structure. Extending our main theorem to all graphs with contractible quotients is left as an open problem.

Motivation

Our ultimate motivation for studying colimits over strong trees (or strong-tree colimits) is to prove a type-theoretic version of the Brown representability theorem. In our proof, we take a set-indexed family $S : I \to \mathcal{U}^*$ of pointed types along with the type $\mathcal{T}(S)$ of all iterated strong-like colimits of diagrams in S. As strong trees are connected, every type belonging to $\mathcal{T}(S)$ is pointed in a natural way.

By taking strong-tree colimits rather than arbitrary ones, we can prove that diagrams of pointed maps satisfy certain coherence conditions involving the maps' proofs of pointedness. This lets us establish the following Yoneda-like lemma, which follows in part from Theorem 1 (stated in the next section).

Lemma 1. *Assume that all function types between elements of $\mathcal{T}(S)$ satisfy Whitehead's theorem. Let A and B be elements of $\mathcal{T}(S)$. For every $(f, f_p) : A \to_* B$, if the map*

$$\|S_x \to_* A\|_0 \xrightarrow{(f,f_p)\circ -} \|S_x \to_* B\|_0$$

is an equivalence for all $x : I$, then f is an equivalence.

This is a key lemma for our anticipated proof of Brown representability, which will say that certain functors $(\mathcal{U}^*)^{\mathrm{op}} \to \mathbf{Set}$ are representable on $\mathcal{T}(S)$.

Strong trees and colimits

A graph Γ is a pair consisting of a type $\Gamma_0 : \mathcal{U}$ of vertices and a type $\Gamma_1(i, j)$ of edges for each $i, j : \Gamma_0$. A diagram over Γ consists of a family $F : \Gamma_0 \to \mathcal{U}$ of types and a function $F_{i,j,g} : F(i) \to F(j)$ for each $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$. The colimit $\mathsf{colim}_\Gamma(F)$ of a diagram F over Γ is the higher inductive type generated by

- $\iota : \prod_{i:\Gamma_0} F(i) \to \mathsf{colim}_\Gamma(F)$
- $\kappa : \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} \prod_{x:F(i)} \iota_j(F_{i,j,g}(x)) = \iota_i(x)$.

Moreover, we can form the (standard) limit of F over Γ:

$$\mathsf{lim}_\Gamma(F) :\equiv \sum_{x:\prod_{i:\Gamma_0} F(i)} \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} F_{i,j,g}(x_i) = x_j.$$

Let Γ be a graph. We want a type consisting of *zigzags of chains of composable edges* in Γ. For us, each zigzag will start at a fixed vertex k, marked by nil^k. Formally, define the indexed inductive type $\mathcal{Z}^k_\Gamma : \Gamma_0 \to \mathcal{U}$ with the constructors

- $\mathsf{nil}^k : \mathcal{Z}^k_\Gamma(k)$
- $\mathsf{right}^k : \prod_{i,j:\Gamma_0} \mathcal{Z}^k_\Gamma(i) \to \Gamma_1(i,j) \to \mathcal{Z}^k_\Gamma(j)$
- $\mathsf{left}^k : \prod_{i,j:\Gamma_0} \mathcal{Z}^k_\Gamma(i) \to \Gamma_1(j,i) \to \mathcal{Z}^k_\Gamma(j)$.

In our work, we care about diagrams of pointed types whose maps are pointed. Suppose that F is a diagram over Γ where each $F(i)$ has a basepoint b_i and each function $F_{i,j,g}$ is equipped with a proof $p_{i,j,g} : F_{i,j,,g}(b_i) = b_j$. Also, suppose that $j_0 : \Gamma_0$, so that $(\mathsf{colim}_\Gamma(F), \iota_{j_0}(b_{j_0}))$ is a pointed type.

Now, let K be a cocone under F, i.e., a type equipped with a family of functions $x : \prod_{i:\Gamma_0} F(i) \to K$ and a coherence condition $r : \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} x_j \circ F_{i,j,g} = x_i$ for x. By induction on $\mathcal{Z}^{j_0}_\Gamma$, we can find a term $\mathsf{ptd}_{(x,r)} : \prod_{i:\Gamma_0} \mathcal{Z}^{j_0}_\Gamma(i) \to (x_i(b_i) = x_{j_0}(b_{j_0}))$. Assume that Γ is equipped with a function $\nu_{j_0,-} : \prod_{i:\Gamma_0} \mathcal{Z}^{j_0}_\Gamma(i)$. For each $i : \Gamma_0$, take

$$\mathsf{pt}^\nu_{(x,r)}(i) :\equiv \mathsf{ptd}_{(x,r)}(i, \nu_{j_0,i}) \;:\; x_i(b_i) = x_{j_0}(b_{j_0}).$$

Thus, as long as each vertex of Γ is connected by a zigzag from j_0, we have a proof of pointedness for each member of the family x of functions.

Definition 1 (Strong tree). We say that Γ is a *strong tree* if for every $i, j : \Gamma_0$ and $g : \Gamma_1(i,j)$, we have a term

$$\sigma_{i,j,g} \;:\; \left(\nu_{j_0,i} = \mathsf{left}^{j_0}_{j,i}(\nu_{j_0,j}, g)\right) + \left(\nu_{j_0,j} = \mathsf{right}^{j_0}_{i,j}(\nu_{j_0,i}, g)\right).$$ [1]

For example, W-types and $\mathbb{Z}$ are strong trees when viewed as graphs.

If Γ is a strong tree, then we have a term

$$\rho(r_{j,i,g}) \;:\; \mathsf{transport}^{k \mapsto k(b_i) = x_{j_0}(b_{j_0})}(r_{j,i,g}, \mathsf{ap}_{x_j}(p_{i,j,g}) \cdot \mathsf{pt}^\nu_{(x,r)}(j)) =_{x_i(b_i) = x_{j_0}(b_{j_0})} \mathsf{pt}^\nu_{(x,r)}(i)$$

for each $j, i : \Gamma_0$ and $g : \Gamma_1(i,j)$. The terms

$$\mathsf{pair}^=(r_{j,i,g}, \rho(r_{j,i,g})) \;:\; \left(x_j, \mathsf{pt}^\nu_{(x,r)}(j)\right) \circ (F_{i,j,g}, p_{i,j,g}) =_{F(i) \to_* K} \left(x_i, \mathsf{pt}^\nu_{(x,r)}(i)\right)$$

together give us a proof of coherence for x as a family of *pointed* maps. This makes (K, x, r) into a cocone under F in the pointed category.

Theorem 1. *Suppose that Γ is a strong tree. For all pointed types T, we have an equivalence*

$$e_{\mathsf{colim}_\Gamma(F), \iota_{j_0}(b_{j_0}), T} : (\mathsf{colim}_\Gamma(F) \to_* T) \xrightarrow{\simeq} \mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(F(i) \to_* T)$$
$$(f, \beta) \mapsto \left(\lambda j.\left(f \circ \iota_j, \mathsf{pt}^\nu_{(\lambda j. f \circ \iota_j, \beta, \lambda j \lambda i \lambda g. \tau_{j,i,g})}(j)\right), \lambda j \lambda i \lambda g. \mathsf{pair}^=(\tau_{j,i,g}, \rho(\tau_{j,i,g}))\right).$$
$$(\tau_{j,i,g} :\equiv \mathsf{ap}_{f \circ -}(\mathsf{funext}(\lambda z. \kappa_{i,j,g}(z))))$$

Moreover, for every pointed type U and map $h^* : T \to_* U$, we have a commutative square

$$\begin{array}{ccc}
(\mathsf{colim}(F) \to_* T) & \xrightarrow{\;h^* \circ -\;} & (\mathsf{colim}(F) \to_* U) \\
\Big\downarrow e_T & & \Big\downarrow e_U \\
\mathsf{lim}(F \to_* T) & \xrightarrow[(x,r) \mapsto \left(\lambda j. h^* \circ x_j, \lambda j \lambda i \lambda g. A_{h^*}(j,i,g) \cdot \mathsf{ap}_{h^* \circ -}(r_{j,i,g})\right)]{} & \mathsf{lim}(F \to_* U)
\end{array}$$

where $A_{h^*}(j,i,g)$, in the bottom arrow, has type $(h^* \circ x_j) \circ (F_{i,j,g}, p_{i,j,g}) = h^* \circ (x_j \circ (F_{i,j,g}, p_{i,j,g}))$.

We also have showed that if (Z, x_Z, r_Z) is a colimiting cocone under F in the pointed category, then Z and $\mathsf{colim}_\Gamma(F)$ are equivalent as pointed types. In this sense, Theorem 1 expresses that colimits over strong trees are preserved by the forgetful functor from pointed types to types. As all strong trees have contractible quotients, this is similar to the classical theorem that the forgetful functor from the ∞-category of pointed spaces to the ∞-category of spaces preserves colimits over weakly contractible ∞-categories.

[1] This condition should imply that $\nu_{j_0,j_0} = \mathsf{nil}^{j_0}$.