# The category of iterative sets in HoTT

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## 1 Motivation

When working in Univalent Foundations, the traditional role of the category **Set** is replaced by the category **HSet** of homotopy sets (h-sets); types with h-propositional identity types. Many of the expected properties of **Set** transfer to **HSet** ((co)completeness, exactness, local cartesian closure etc.). Notably, however, Ob(**HSet**), is not itself an h-set for two reasons:

1. The type of all h-sets is large.
2. The type of all h-sets has non-h-propositional identity types.

The first problem is well-studied and analagous to how Ob(**Set**) is a proper class. One can stratify h-sets in parallel with types into a hierarchy $\mathcal{HS}et_i := \sum_{X:\mathcal{U}_i} \text{is-h-set } X$. The second problem emerges only in Univalent Foundations and is more persistent; in fact, any univalent category containing non-trivial automorphisms cannot have an h-set of objects. For many purposes, one can replace a univalent category by a suitably equivalent pre-category with a set of objects—a set category—but it is far from obvious that such a replacement for $\mathbf{HSet}_i$ should exist.

In this work, we equip the type of iterative sets ($\mathcal{V}_i$) (relative to a universe $\mathcal{U}_i$) due to Gylterud [4] with the structure of a Tarski universe and argue that it serves as an adequate replacement for h-sets. We give a new formalisation of $\mathcal{V}_i$, show that it organizes into a set category $\mathbf{V}_i$ and construct a model of type theory upon it, overcoming previous challenges of doing the same with $\mathbf{HSet}_i$.

## 2 $\mathcal{V}_i$ as a Tarski universe

Following the idea of Aczel [1] and later Gylterud [4], we define $\mathcal{V}_i$ as the subtype of $\mathcal{V}_i^\infty := W_{A:\mathcal{U}_i} A$ consisting of the hereditary embeddings. More specifically, we specify the following predicate on $\mathcal{V}_i^\infty$:

$$\begin{aligned} &\text{iterative-set} : \mathcal{V}_i^\infty \to \mathcal{U}_i \\ &\text{iterative-set}\,(\sup\ A\ f) := \text{is-embedding } f \times \prod_{a:A} \text{iterative-set}\,(f\,a) \end{aligned}$$

We then define $\mathcal{V}_i$ as $\sum_{v:\mathcal{V}_i^\infty} \text{iterative-set } v$. Even though $\mathcal{V}_i^\infty$ has the same h-level as $\mathcal{U}_i$, Gylterud has shown that $\mathcal{V}_i$ is an h-set [4]. In order to use $\mathcal{V}_i$ as a Tarski universe, we must equip it with a decoding function $\text{El} : \mathcal{V}_i \to \mathcal{U}_i$ allowing us to regard an element of $\mathcal{V}_i$ as a type:

$$\text{El}\,(\sup\ A\ f,\ p) := A$$

It remains to show that $\mathcal{V}_i$ is closed under the usual type-formers. For instance, we define $\Pi : \prod_{A:\mathcal{V}_i}(\text{El}\ \ A \to \mathcal{V}_i) \to \mathcal{V}_i$ along with a path $\text{El}\,(\Pi\, A\, B) = \prod_{a:\text{El}\ A} \text{El}\,(B\, a)$. In fact, the computational properties of $\mathcal{V}_i$ ensure that $\text{El}\,(\Pi\, A\, B) \equiv \prod_{a:\text{El}\ A} \text{El}\,(B\, a)$ holds definitionally. The same is true for all the usual type-formers, which makes it highly ergonomic to work with $\mathcal{V}_i$.

**Theorem 1.** *The Tarski universe* ($\mathcal{V}_i$, El) *is closed under dependent sums, dependent products, identity types, and quotients, and contains the unit type, the empty type, and the natural numbers type.*

For any $A : \mathcal{V}_i$, by construction, El $A$ embeds into $\mathcal{V}_i$. Since $\mathcal{V}_i$ is an h-set, it follows that El $A$ is an h-set. Accordingly, $\mathcal{V}_i$ serves as an h-set of h-sets.

## 3 Category structure on $\mathcal{V}_i$

We equip $\mathcal{V}_i$ with a set category structure $\mathbf{V}_i$ by taking $\hom(A, B) :\equiv \mathrm{El}\ A \to \mathrm{El}\ B$. By Theorem 1, $\mathbf{V}_i$ enjoys a number of categorical properties. In particular, we have formalised a proof that it is a finitely cocomplete locally cartesian closed category. There is a natural forgetful functor $U : \mathbf{V}_i \to \mathbf{HSet}_i$ which respects all of the aforementioned properties.

What, precisely, is the relationship between $\mathbf{HSet}_i$ and $\mathbf{V}_i$? The forgetful functor $U$ is fully-faithful, but in general not essentially surjective. Since $\mathbf{HSet}_i$ is univalent, the question boils down to whether the object part of $U$, namely $\mathrm{El} : \mathcal{V}_i \to \mathcal{U}_i$, is surjective. If so, the inclusion of $\mathbf{V}_i$ into $\mathbf{HSet}_i$ is an equivalence of precategories. This property is what Shulman [8] calls the axiom of well-founded materialization, and would follow from AC.

**Theorem 2.** *In the presence of choice, $U$ is essentially surjective and therefore witnesses $\mathbf{HSet}_i$ as the* Rezk completion *of $\mathbf{V}_i$.*

It therefore follows that—in the presence of choice—all categorical properties of $\mathbf{HSet}_i$ transfer to $\mathbf{V}_i$ and vice versa. Note, however, that our results demonstrate that $\mathbf{V}_i$ is well-behaved even without choice.

## 4 A model of type theory in $\mathbf{V}_i$

Surprisingly, even though **HSet** is locally cartesian closed, it is non-obvious how to equip it with a category with families (CwF) structure. The obstruction lies in the homotopy level of $\mathcal{HS}et_i$; as a family of 1-types, we cannot regard $\mathrm{Ob}(\mathbf{HSet}/-)$ as a presheaf. We stress that this is independent of the normal strictness issues involved in constructing a model of type theory. Rather, it is a manifestation of the fact that the collection of h-sets is a proper h-groupoid.

The type of objects in $\mathbf{V}_i$, on the other hand, is an h-set. Thus, we can equip it with a CwF structure analogous to the standard one for **Set** [6], and prove that it supports the usual type-formers.

## 5 Related work

There have been various constructions of iterative hierarchies in structural settings, such as type theory or category theory, before. Some notable ones are Fourman–Hayashi [3, 5] in topos theory, and Aczel's setoid based model [1] and the HoTT book's $V$ [9] in type theory. Shulman's work [8] on the category of material sets one obtains from Fourman—Hayashi , resembles closely what we get for $\mathbf{V}_i$. One added nuance, which we get from working in HoTT, is the perspective of $\mathbf{HSet}_i$ as the Rezk-completion of $\mathbf{V}_i$.

The type $\mathcal{V}_i$ is equivalent to the type $V$ in the HoTT book [4], but is in many ways easier to work with as it is not a higher inductive type. In an interesting recent development, de Jong et. al. [2] have shown that the ordinals in $V$ coincide with the type theoretic ordinals. We apply similiar techniques to theirs in order to connect our work to Shulman's axiom of well-founded materialization.

Our work is currently being formalised in the Agda proof assistant, using the agda-unimath library [7]. The formalisation is available at: `https://git.app.uib.no/hott/hott-set-theory`

## References


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