Higher Pro-arrows:  
Towards a Model for Naturality Pretype Theory  
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**Naturality Type Theory**  
Fourty years ago, Reynolds [Rey83] formulated his model of relational parametricity, which was later reorganized as a model of type theory in reflexive graphs [Atk12, AGJ14]. Relational parametricity differs from the categorical notion of naturality in two important ways. First, it is formulated against the action of type operations on relations rather than on functions, and secondly, it is formulated in terms of reflexive graphs, which are exactly categories with identities but without composition. Why is one approach much more popular in category theory, and the other in functional programming?

Category theorists usually do not define ‘naturality’ in general; rather they have the discipline to use specific notions such as natural, dinatural or extranatural transformations, limits and colimits, ends and co-ends, … For practical functional programming, this is unworkable: only very specific function types can be seen as natural transformation types, whereas dinatural and extranatural transformations do not (always) compose. The underlying problem is that of variance: many interesting type operations are not covariant. By asking good behaviour w.r.t. relations instead of functions, variance becomes unimportant, making parametricity a much more robust property. Since not all type formers preserve composition of relations, the composition operation is dropped, leaving us with a reflexive graph model.

Another way to get rid of variance is by considering only invertible functions, which yields HoTT [Uni13]. Both approaches are unsatisfactory: where HoTT fails to consider non-invertible functions, a relational approach forgets their computational behaviour. In the ideal scenario, we would have a type system that allows us to use each of the approaches, while relating them: HoTT when functions are invertible, brittle functoriality and naturality when possible, robust but non-computational parametricity when necessary. We call such an envisioned system **naturality type theory**. A first proposal for such a system, justified merely by intuitive arguments, was made in [Nuy15]. An example problem showing the need for such a system, is elaborated in [Nuy20, §1.2].

**Fibrancy**  
HoTT [BCH14, CMS20, CCHM17, Hub16, KLV12], parametricity [AGJ14, BCM15, NVD17, ND18], directed type theory and combinations [CH20, RS17, WL20] are typically modeled in presheaf categories. However, operations such as composition of and transport along any sort of ‘lines’ cannot be prescribed by morphisms in a presheaf’s base category and are instead added by imposing a notion of algebraic fibrancy on types. Examples of such notions are: Kan types (allowing composition of and transport along paths) [KLV12, BCH14, CCHM17], covariant types (transport along morphisms) [RS17], Segal types (composition of morphisms) [RS17], Rezk types (path-isomorphism univalence) [RS17], and (bridge)-discrete types [AGJ14, NVD17, CH20] (which satisfy Reynolds’ identity extension lemma). Many type systems only include fibrant types. In directed type theory, this is infeasible as, unsurprisingly given the discussion above, the relevant notions of Segal fibrancy and covariance are brittle. Types that are not fibrant are often called **pre** types. In this first step, we do not yet try to formally define and model the relevant notions of fibrancy (although of course we keep them in mind as they are the very purpose of the system); as such the title mentions a ‘pretype theory’.

**The Type System**  
Although it may be unclear what it means to build a model without specifying a type system, in this work we solely focus on the model, as the type system then materializes by established or emerging techniques. Indeed, any constellation of presheaf categories with adjoint sequences of functors between them, automatically gives rise to dependent right adjoints (DRAs, [BCM+20]) by [Nuy20, lemma 5.2.3 and thm. 6.4.1], and thus constitutes a model of multimode
type theory (MTT, [GKNB21]). A choice of base category functor whose left Kan extension models context extension with a directed interval variable, yields a model of the modal transpension system (MTraS) [ND21], which deals with substructural shape variables and internalizes crucial characteristics of presheaf models. The interaction of MTraS’s (co)quantification modalities with pre-existing modalities is discussed in the associated technical report [Nuy21].

The Model The model we propose, can be invented in three ways, with the exact same result, which we sketch below after introducing some prerequisites.

*n-fold categories.* A double category consists of objects, horizontal arrows, vertical arrows, and squares, with horizontal and vertical identities and composition operations. This notion can be straightforwardly generalized to n-fold categories. These can be defined as n-fold simplicial sets, i.e. presheaves over the simplex category $\Delta^n$, satisfying the Segal condition in each dimension. Since it is a notion of fibrancy, we disregard the Segal condition.

Twisted cube category. The twisted cube category $\boxtimes$ [PK19] is a subcategory of the category of non-empty finite linear orders (of which $\Delta$ is a skeleton) whose objects are generated by $\{\ast\}$ and the twisted prism functor $\sqcup \times I : W \mapsto W^{op} \cup W$, where $\cup$ takes the coproduct of two posets and then declares the elements from the left less than those from the right. We prefer to use $\boxtimes$ over $\Delta$ because it seems to interact better with cubical models of HoTT and parametricity and because $\sqcup \times I$ can be studied using MTraS [ND21]. So we will view n-fold categories as n-fold twisted cubical sets, i.e. presheaves over $\boxtimes^n$.

Approach 1: Higher pro-arrow pre-equipments. Set and Cat are well-known to have the structure of a category, but if we consider not just arrows (functions/functors) but also pro-arrows (relations/profunctors), they get the richer structure of a pro-arrow equipment. A pro-arrow equipment or just equipment [nLa20] is a double category with extra structure. The arrow classes are called arrows ($\to$) and pro-arrows ($\nrightarrow$), and the structure ensures that every arrow $\varphi : x \to y$ has a companion $\hat{\varphi} : x \nrightarrow y$ and a conjoint $\check{\varphi} : y \nrightarrow x$, together with squares expressing that these pro-arrows behave as the graph relations/profunctors of $\varphi$. The equipment of equipments $\mathbb{Eqmnt}$ in turn has further structure: we can consider not only arrows (functors) and pro-arrows (profunctors, bearing a notion of heterogeneous arrows) but also pro-pro-arrows (pro-profunctors, bearing a notion of heterogeneous pro-arrows). This asks for a notion of higher equipments, to be modelled in n-fold twisted cubical sets, where we interpret arrows of dimension $i \geq 1$ as pro$^{i-1}$-arrows, an unpractical word that we replace with $i$-jets. In general, $(i+1)$-jets $J : A \rightarrow_{i+1} B$ between types will be containers for heterogeneous $i$-jets $j : a \rightarrow^i b$ between their elements. Viewing type morphisms $A \rightarrow B$ as containers for mappings $a \mapsto b$, we can introduce $\rightarrow^i_0$ as the mapping interval. Equipment-natural functions $f : (\mathbf{nat} \times_1 A) \rightarrow B x$ are then functions sending $(i+1)$-jets $\bar{x} : x_1 \rightarrow_{i+1} x_2$ to $i$-jets $f \bar{x} : f x_1 \rightarrow f x_2$ along the $(i+1)$-jet $B \bar{x} : B x_1 \rightarrow_{i+1} B x_2$. General modalities will specify that $i$-jets are sent to $j$-jets co-/contra-/isovariantly.

Approach 2: Directing Degrees of Relatedness In [Nuy20, ch. 9], I discuss how the reflexive graph model of parametricity [AGJ14] further evolved into a cubical model [BCM15], and from there to a modal [NVD17] and even multimode [ND18] system in which every type is equipped with a number of relations (‘degrees of relatedness’) mediated by the modalities. This system is essentially a non-directed version of what is proposed above, modelled in n-fold cubical sets, and already justifies the higher structure that we propose in a purely relational setting.

Approach 3: From Tamsamani and Simpson’s model of higher categories Cheng and Lauda [CL04] accessibly explain how Tamsamani [Tam99] and Simpson [Sim11] define n-categories as n-fold categories in which certain cells are required to be trivial. This model can be heterogenized by lifting the triviality condition, leading again to the same ideas.

Current Status As explained above, we have identified the necessary building blocks to build this model, and it is generally clear how to fit them together. What remains to be done is to sort out the details of the base categories and the modalities, and the properties of the twisted interval. This is not a matter of success or failure, but a matter of considering different options and analyzing their pros and cons. It is unlikely that one solution will be the panacea.
Acknowledgements  I want to thank Andreas Abel, Thorsten Altenkirch, Paolo Capriotti, Jesper Cockx, Dominique Devriese, Simon Huber, Nicolai Kraus, Dan Licata, Gun Pinyo, Josselin Poirot, Michael Shulman, Andrea Vezzosi, and others who I am disgracefully forgetting, for related discussions over the years. I thank the anonymous reviewers and Jonathan Weinberger for their well-defined and constructive feedback, which has improved the abstract.

Andreas Nuyts holds a Postdoctoral Fellowship from the Research Foundation - Flanders (FWO). This research is partially funded by the Research Fund KU Leuven.

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