Higher Pro-arrows: Towards a Model for Naturality Pretype Theory

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Naturality Type Theory Fourty years ago, Reynolds [Rey83] formulated his model of relational parametricity, which was later reorganized as a model of type theory in reflexive graphs [Atk12, AGJ14]. Relational parametricity differs from the categorical notion of naturality in two important ways. First, it is formulated against the action of type operations on relations rather than on functions, and secondly, it is formulated in terms of reflexive graphs, which are exactly categories with identities but without composition. Why is one approach much more popular in category theory, and the other in functional programming?

Category theorists usually do not define 'naturality' in general; rather they have the discipline to use specific notions such as natural, dinatural or extranatural transformations, limits and colimits, ends and co-ends, ... For practical functional programming, this is unworkable: only very specific function types can be seen as natural transformation types, whereas dinatural and extranatural transformations do not (always) compose. The underlying problem is that of variance: many interesting type operations are not covariant. By asking good behaviour w.r.t. relations instead of functions, variance becomes unimportant, making parametricity a much more robust property. Since not all type formers preserve composition of relations, the composition operation is dropped, leaving us with a reflexive graph model.

Another way to get rid of variance is by considering only invertible functions, which yields HoTT [Uni13]. Both approaches are unsatisfactory: where HoTT fails to consider non-invertible functions, a relational approach forgets their computational behaviour. In the ideal scenario, we would have a type system that allows us to use each of the approaches, while relating them: HoTT when functions are invertible, brittle functoriality and naturality when possible, robust but non-computational parametricity when necessary. We call such an envisioned system **naturality type theory**. A first proposal for such a system, justified merely by intuitive arguments, was made in [Nuy15]. An example problem showing the need for such a system, is elaborated in [Nuy20, §1.2].

Fibrancy HoTT [BCH14, CMS20, CCHM17, Hub16, KLV12], parametricity [AGJ14, BCM15, NVD17, ND18], directed type theory and combinations [CH20, RS17, WL20] are typically modelled in presheaf categories. However, operations such as composition of and transport along any sort of 'lines' cannot be prescribed by morphisms in a presheaf's base category and are instead added by imposing a notion of algebraic fibrancy on types. Examples of such notions are: Kan types (allowing composition of and transport along paths) [KLV12, BCH14, CCHM17], covariant types (transport along morphisms) [RS17], Segal types (composition of morphisms) [RS17], Rezk types (path-isomorphism univalence) [RS17], and (bridge)-discrete types [AGJ14, NVD17, CH20] (which satisfy Reynolds' identity extension lemma). Many type systems only include fibrant types. In directed type theory, this is infeasible as, unsurprisingly given the discussion above, the relevant notions of Segal fibrancy and covariance are brittle. Types that are not fibrant are often called *pre*types. In this first step, we do not yet try to formally define and model the relevant notions of fibrancy (although of course we keep them in mind as they are the very purpose of the system); as such the title mentions a '*pre*type theory'.

The Type System Although it may be unclear what it means to build a model without specifying a type system, in this work we solely focus on the model, as the type system then materializes by established or emerging techniques. Indeed, any constellation of presheaf categories with adjoint sequences of functors between them, automatically gives rise to dependent right adjoints (DRAs, [BCM^{+}20]) by [Nuy20, lemma 5.2.3 and thm. 6.4.1], and thus constitutes a model of multimode type theory (MTT, [GKNB21]). A choice of base category functor whose left Kan extension models context extension with a directed interval variable, yields a model of the modal transpension system (MTraS) [ND21], which deals with substructural shape variables and internalizes crucial characteristics of presheaf models. The interaction of MTraS's (co)quantification modalities with pre-existing modalities is discussed in the associated technical report [Nuy21].

The Model The model we propose, can be invented in three ways, with the exact same result, which we sketch below after introducing some prerequisites.

***n*-fold categories.** A *double category* consists of objects, horizontal arrows, vertical arrows, and squares, with horizontal and vertical identities and composition operations. This notion can be straightforwardly generalized to n-fold categories. These can be defined as n-fold simplicial sets, i.e. presheaves over the simplex category Δ^n, satisfying the Segal condition in each dimension. Since it is a notion of fibrancy, we disregard the Segal condition.

Twisted cube category. The twisted cube category $\bowtie$ [PK19] is a subcategory of the category of non-empty finite linear orders (of which Δ is a skeleton) whose objects are generated by $\{*\}$ and the *twisted prism functor* $\sqcup \ltimes \mathbb{I} : W \mapsto W^{\mathsf{op}} \uplus_{<} W$, where $\uplus_{<}$ takes the coproduct of two posets and then declares the elements from the left less than those from the right. We prefer to use $\bowtie$ over Δ because it seems to interact better with cubical models of HoTT and parametricity and because $\sqcup \ltimes \mathbb{I}$ can be studied using MTraS [ND21]. So we will view n-fold categories as n-fold **twisted cubical sets**, i.e. presheaves over $\bowtie^n$.

Approach 1: Higher pro-arrow pre-equipments. Set and Cat are well-known to have the structure of a category, but if we consider not just arrows (functions/functors) but also pro-arrows (relations/profunctors), they get the richer structure of a pro-arrow equipment. A *pro-arrow equipment* or just *equipment* [nLa20] is a double category with extra structure. The arrow classes are called *arrows* ($\to$) and *pro-arrows* ($\nrightarrow$), and the structure ensures that every arrow $\varphi : x \to y$ has a *companion* $\hat{\varphi} : x \nrightarrow y$ and a *conjoint* $\check{\varphi} : y \nrightarrow x$, together with squares expressing that these pro-arrows behave as the graph relations/profunctors of φ. The equipment of equipments Eqmnt in turn has *further* structure: we can consider not only arrows (functors) and pro-arrows (profunctors, bearing a notion of heterogeneous arrows) but also pro-pro-arrows (pro-profunctors, bearing a notion of heterogeneous pro-arrows). This asks for a notion of higher equipments, to be modelled in n-fold twisted cubical sets, where we interpret arrows of dimension $i \geq 1$ as pro^{i-1}-arrows, an unpractical word that we replace with ***i*-jets**. In general, $(i+1)$-jets $J : A \curvearrowright_{i+1} B$ between types will be containers for heterogeneous i-jets $j : a \curvearrowright_i^J b$ between their elements. Viewing type morphisms $A \to B$ as containers for mappings $a \mapsto b$, we can introduce $\mathbb{J}_0$ as the mapping interval. Equipment-natural functions $f : (\mathbf{nat} \shortmid x : A) \to B\,x$ are then functions sending $(i+1)$-jets $\vec{x} : x_1 \curvearrowright_{i+1}^A x_2$ to i-jets $f\,\vec{x} : f\,x_1 \curvearrowright_i^{B\,\vec{x}} f\,x_2$ along the $(i+1)$-jet $B\,\vec{x} : B\,x_1 \curvearrowright_{i+1}^{\mathsf{U}} B\,x_2$. General modalities will specify that i-jets are sent to j-jets co-/contra-/isovariantly.

Approach 2: Directifying Degrees of Relatedness In [Nuy20, ch. 9], I discuss how the reflexive graph model of parametricity [AGJ14] further evolved into a cubical model [BCM15], and from there to a modal [NVD17] and even multimode [ND18] system in which every type is equipped with a number of relations ('degrees of relatedness') mediated by the modalities. This system is essentially a non-directed version of what is proposed above, modelled in n-fold cubical sets, and already justifies the higher structure that we propose in a purely relational setting.

Approach 3: From Tamsamani and Simpson's model of higher categories Cheng and Lauda [CL04] accessibly explain how Tamsamani [Tam99] and Simpson [Sim11] define n-categories as n-fold categories in which certain cells are required to be trivial. This model can be heterogenized by lifting the triviality condition, leading again to the same ideas.

Current Status As explained above, we have identified the necessary building blocks to build this model, and it is generally clear how to fit them together. What remains to be done is to sort out the details of the base categories and the modalities, and the properties of the twisted interval. This is not a matter of success or failure, but a matter of considering different options and analyzing their pros and contras. It is unlikely that one solution will be the panacea.

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References

[AGJ14] Robert Atkey, Neil Ghani, and Patricia Johann. A relationally parametric model of dependent type theory. In *Principles of Programming Languages*, 2014. doi:10.1145/2535838.2535852.

[Atk12] Robert Atkey. Relational parametricity for higher kinds. In *Computer Science Logic (CSL'12) - 26th International Workshop/21st Annual Conference of the EACSL*, volume 16 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 46–61, 2012. doi:10.4230/LIPIcs.CSL.2012.46.

[BCH14] Marc Bezem, Thierry Coquand, and Simon Huber. A Model of Type Theory in Cubical Sets. In *19th International Conference on Types for Proofs and Programs (TYPES 2013)*, volume 26, pages 107–128, Dagstuhl, Germany, 2014. URL: http://drops.dagstuhl.de/opus/volltexte/2014/4628, doi:10.4230/LIPIcs.TYPES.2013.107.

[BCM15] Jean-Philippe Bernardy, Thierry Coquand, and Guilhem Moulin. A presheaf model of parametric type theory. *Electron. Notes in Theor. Comput. Sci.*, 319:67 – 82, 2015. doi:http://dx.doi.org/10.1016/j.entcs.2015.12.006.

[BCM+20] Lars Birkedal, Ranald Clouston, Bassel Mannaa, Rasmus Ejlers Møgelberg, Andrew M. Pitts, and Bas Spitters. Modal dependent type theory and dependent right adjoints. *Mathematical Structures in Computer Science*, 30(2):118–138, 2020. doi:10.1017/S0960129519000197.

[CCHM17] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: A constructive interpretation of the univalence axiom. *FLAP*, 4(10):3127–3170, 2017. URL: http://www.cse.chalmers.se/~simonhu/papers/cubicaltt.pdf.

[CH20] Evan Cavallo and Robert Harper. Internal parametricity for cubical type theory. In *28th EACSL Annual Conference on Computer Science Logic, CSL 2020, January 13-16, 2020, Barcelona, Spain*, pages 13:1–13:17, 2020. doi:10.4230/LIPIcs.CSL.2020.13.

[CL04] Eugenia Cheng and Aaron Lauda. Higher-dimensional categories: an illustrated guide book. 2004. URL: http://eugeniacheng.com/guidebook/.

[CMS20] Evan Cavallo, Anders Mörtberg, and Andrew W Swan. Unifying Cubical Models of Univalent Type Theory. In Maribel Fernández and Anca Muscholl, editors, *Computer Science Logic (CSL 2020)*, volume 152 of *LIPIcs*, pages 14:1–14:17, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. URL: https://drops.dagstuhl.de/opus/volltexte/2020/11657, doi:10.4230/LIPIcs.CSL.2020.14.

[GKNB21] Daniel Gratzer, G. A. Kavvos, Andreas Nuyts, and Lars Birkedal. Multimodal dependent type theory. *Log. Methods Comput. Sci.*, 17(3), 2021. doi:10.46298/lmcs-17(3:11)2021.

[Hub16] Simon Huber. *Cubical Interpretations of Type Theory.* PhD thesis, University of Gothenburg, Sweden, 2016. URL: http://www.cse.chalmers.se/~simonhu/misc/thesis.pdf.

[KLV12] Chris Kapulkin, Peter LeFanu Lumsdaine, and Vladimir Voevodsky. The simplicial model of univalent foundations. 2012. Preprint, http://arxiv.org/abs/1211.2851.

[ND18] Andreas Nuyts and Dominique Devriese. Degrees of relatedness: A unified framework for parametricity, irrelevance, ad hoc polymorphism, intersections, unions and algebra in dependent type theory. In *Logic in Computer Science (LICS) 2018, Oxford, UK, July 09-12, 2018*, pages 779–788, 2018. doi:10.1145/3209108.3209119.

[ND21] Andreas Nuyts and Dominique Devriese. Transpension: The right adjoint to the pi-type, 2021. arXiv:2008.08533.

[nLa20] nLab authors. 2-category equipped with proarrows, April 2020. Revision 32. URL: http://ncatlab.org/nlab/show/2-category%20equipped%20with%20proarrows.

[Nuy15] Andreas Nuyts. Towards a directed homotopy type theory based on 4 kinds of variance. Master's thesis, KU Leuven, Belgium, 2015. URL: https://anuyts.github.io/files/mathesis.pdf.

[Nuy20] Andreas Nuyts. *Contributions to Multimode and Presheaf Type Theory*. PhD thesis, KU Leuven, Belgium, 8 2020. URL: https://lirias.kuleuven.be/3065223.

[Nuy21] Andreas Nuyts. The transpension type: Technical report, 2021. arXiv:2008.08530.

[NVD17] Andreas Nuyts, Andrea Vezzosi, and Dominique Devriese. Parametric quantifiers for dependent type theory. *PACMPL*, 1(ICFP):32:1–32:29, 2017. URL: http://doi.acm.org/10.1145/3110276, doi:10.1145/3110276.

[PK19] Gun Pinyo and Nicolai Kraus. From cubes to twisted cubes via graph morphisms in type theory. *CoRR*, abs/1902.10820, 2019. URL: http://arxiv.org/abs/1902.10820, arXiv:1902.10820.

[Rey83] John C. Reynolds. Types, abstraction and parametric polymorphism. In *IFIP Congress*, pages 513–523, 1983.

[RS17] E. Riehl and M. Shulman. A type theory for synthetic ∞-categories. *ArXiv e-prints*, May 2017. arXiv:1705.07442.

[Sim11] Carlos Simpson. *Homotopy Theory of Higher Categories: From Segal Categories to n-Categories and Beyond*, volume 19. Cambridge University Press, 2011.

[Tam99] Zouhair Tamsamani. Sur des notions de n-categorie et n-groupoide non strictes via des ensembles multi-simpliciaux (on the notions of a nonstrict n-category and n-groupoid via multi-simplicial sets). *K-theory*, 16(1):51–99, 1999.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. http://homotopytypetheory.org/book, IAS, 2013.

[WL20] Matthew Z. Weaver and Daniel R. Licata. A constructive model of directed univalence in bicubical sets. In Holger Hermanns, Lijun Zhang, Naoki Kobayashi, and Dale Miller, editors, *LICS '20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science, Saarbrücken, Germany, July 8-11, 2020*, pages 915–928. ACM, 2020. doi:10.1145/3373718.3394794.