Internal languages of diagrams of ∞-toposes

Taichi Uemura

July 29, 2022

Abstract

Homotopy type theory (HoTT) is a handy language for reasoning about an ∞-topos. However, sometimes we have more than one ∞-toposes related to each other by some functors, in which case plain HoTT is not sufficiently rich because the actions of the functors are not internalized.

In this talk, we consider a certain class of diagrams of ∞-toposes for which plain HoTT remains a sufficiently rich internal language. We show that a special form of an inverse diagram of ∞-toposes is reconstructed internally to its oplax limit via lex modalities. Then plain HoTT as an internal language of the oplax limit can be used for reasoning about the original diagram.

1 Introduction

The goal of this note is as follows. Let I be a 2-category and $\mathcal{X}$ an I-indexed ∞-topos. When I is nice, $\mathcal{X}$ is reconstructed in an *internal language* of its *oplax limit*.

Homotopy type theory (HoTT) [16] is an internal language of ∞*-toposes* [9]. However, it is not sufficiently rich for reasoning about *diagrams* of ∞-toposes, because we need external reasoning to apply functors and natural transformations in the diagram.

Although proper extension of HoTT would be necessary for internal languages of general diagrams of ∞-toposes, plain HoTT is in some sense sufficient for some special shapes of diagrams. A typical example is a diagram consisting of two ∞-toposes and a lex, accessible functor between them in one direction. The two ∞-toposes are subtoposes of another ∞-topos obtained by the *Artin gluing*, and the functor is reconstructed by composing the inclusion from one subtopos and the reflector to the other. Moreover, this reconstruction is *internal* to the glued ∞-topos, because subtoposes of an ∞-topos correspond to *lex, accessible modalities* in its internal language. Hence, plain HoTT as an internal language of the glued ∞-topos is sufficient to reason about the original diagram.

In this note, we generalize the observation in the previous paragraph. We explain that a special form of *inverse diagram* of ∞-toposes and lex, accessible functors between them is reconstructed internally to its *oplax limit* via lex, accessible modalities.

1.1 Background and related work

This work is much influenced by Sterling's *synthetic Tait computability* [14] which uses modalities in extensional type theory as an internal language of the glued 1-topos for a given functor between 1-toposes. This work is an ∞-analogue and a generalization to more complex diagrams of ∞-toposes. In fact, HoTT and ∞-toposes are more natural setting for lex, accessible modalities than extensional type theory and 1-toposes. The generalization from the Artin gluing to oplax limits of inverse diagrams is inspired by work by Shulman [13].

Sterling's synthetic Tait computability is used for proving positive results about type theories such as canonicity and normalization [15, 6]. A purpose of this work is to develop a technique for proving similar positive results about ∞-*type theories* introduced by Nguyen and the author [10]; see [17] for some ideas. We need a generalization of the Artin gluing to handle diagrams induced by *relative induction principles* of Bocquet, Kaposi, and Sattler [3].

Modalities in HoTT and ∞-toposes are extensively studied by Rijke, Shulman, and Spitters [11] and Anel, Biedermann, Finster, and Joyal [1], respectively. See also [5, 4, 18].

The fact that HoTT is an internal language of ∞-toposes is proved by rectifying ∞-toposes. Shulman [12] showed that every ∞-topos is presented by a well-behaved model category. The interpretation of HoTT in well-behaved model categories is given by Arndt and Kapulkin [2] and Shulman [13], for example.

Another possible approach to internal languages of diagrams of ∞-toposes is to extend HoTT in a similar way to *multimodal type theory* [7] or extend multimodal type theory by univalence and higher inductive types. In this approach we could handle more general diagrams of ∞-toposes, in particular adjunctions between ∞-toposes and (co)monads on ∞-toposes. However, more work is needed to interpret multimodal type theory in diagrams of ∞-toposes because we have to rectify not only ∞-toposes but also functors and natural transformations between them.

2 Modalities in HoTT

We recall the definition of *modalities* in HoTT studied by Rijke, Shulman, and Spitters [11]. We work in HoTT.

Definition 2.1. A *modality* $\mathfrak{m}$ is a predicate $\mathsf{In}_{\mathfrak{m}} : \mathcal{U} \to \mathsf{Prop}$ satisfying the following axioms. Here, we define $\mathcal{U}_{\mathfrak{m}} \equiv \{A : \mathcal{U} \mid \mathsf{In}_{\mathfrak{m}}(A)\}$.

1. $\mathsf{In}_{\mathfrak{m}}$ is a *reflective subuniverse*: it has a *reflector* $\Box_{\mathfrak{m}} : \mathcal{U} \to \mathcal{U}_{\mathfrak{m}}$ and a *unit* $\eta_{\mathfrak{m}} : \prod_{A:\mathcal{U}} A \to \Box_{\mathfrak{m}} A$ satisfying that the precomposition function

$$\lambda f. f \circ \eta_{\mathfrak{m}}(A) : (\Box_{\mathfrak{m}} A \to B) \to (A \to B)$$

   is an equivalence for any $A : \mathcal{U}$ and $B : \mathcal{U}_{\mathfrak{m}}$.

2. $\mathsf{In}_{\mathfrak{m}}$ is closed under dependent pair types.

A type satisfying $\mathsf{In}_{\mathfrak{m}}$ is called $\mathfrak{m}$-*modal*.

Definition 2.2. A modality $\mathfrak{m}$ is *accessible* if it is "presented by small data".

Definition 2.3. A modality $\mathfrak{m}$ is *lex* if $\Box_{\mathfrak{m}}$ preserves finite limits.

Definition 2.4. LAM is an acronym for lex, accessible modality.

Theorem 2.5. *Let $\mathcal{X}$ be an ∞-topos. The LAMs in the internal language of $\mathcal{X}$ bijectively correspond to the subtoposes of $\mathcal{X}$.*

Remark 2.6. Modalities in ∞-toposes are extensively studied by Anel, Biedermann, Finster, and Joyal [1]. Thanks to their results, the proof of Theorem 2.5 is almost straightforward. One non-trivial gap is the difference between type-theoretic accessibility and category-theoretic accessibility.

3 Internal diagrams induced by modalities

Working in HoTT, we consider postulating some LAMs to encode a certain diagram of subuniverses. The fundamental observation is that every pair of LAMs induces a canonical functor between the subuniverses associated to them.

Construction 3.1. Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs. We define a function $\Box^{\mathfrak{m}}_{\mathfrak{n}} : \mathcal{U}_{\mathfrak{m}} \to \mathcal{U}_{\mathfrak{n}}$ to be the restriction of $\Box_{\mathfrak{n}}$ to $\mathcal{U}_{\mathfrak{m}} \subset \mathcal{U}$. It is a "functor" and preserves finite limits.

Remark 3.2. Because defining the type of $(\infty, 1)$-categories in HoTT is an open problem, we do not know how to state that $\Box^{\mathfrak{m}}_{\mathfrak{n}}$ is a functor in HoTT. Nevertheless, we think of $\Box^{\mathfrak{m}}_{\mathfrak{n}}$ as a functor because we can construct the action of $\Box^{\mathfrak{m}}_{\mathfrak{n}}$ on morphisms and prove every instance of coherence laws when needed.

We have two functors $\Box^{\mathfrak{m}}_{\mathfrak{n}} : \mathcal{U}_{\mathfrak{m}} \to \mathcal{U}_{\mathfrak{n}}$ and $\Box^{\mathfrak{n}}_{\mathfrak{m}} : \mathcal{U}_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$ for every pair of LAMs $\mathfrak{m}$ and $\mathfrak{n}$, but we are often interested in only one direction. It is thus useful to cut off one direction.

Definition 3.3. Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs. We write $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$ and say $\mathfrak{m}$ is *left orthogonal to* $\mathfrak{n}$ if $\Box_{\mathfrak{n}}$ takes $\mathfrak{m}$-modal types to contractible types.

Remark 3.4. In [11], $\mathfrak{n}$ is said to be *strongly disjoint from* $\mathfrak{m}$ if $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$.

If $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$, then $\Box^{\mathfrak{m}}_{\mathfrak{n}}$ becomes constant at the unit type. The other direction $\Box^{\mathfrak{n}}_{\mathfrak{m}} : \mathcal{U}_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$ remains non-trivial. Therefore, a pair $(\mathfrak{m}, \mathfrak{n})$ of LAMs such that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$ encodes a diagram consisting of two ∞-toposes and a functor between them in one direction. When both $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$ and $\mathfrak{n} \leq {}^{\perp}\mathfrak{m}$ are assumed, $\mathcal{U}_{\mathfrak{m}}$ and $\mathcal{U}_{\mathfrak{n}}$ are considered unrelated.

Given more than two LAMs, we have canonical natural transformations between the canonical functors.

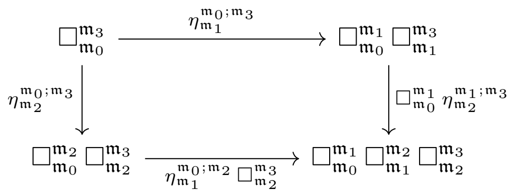
Construction 3.5. Let $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2$ be LAMs. We define

$$\eta^{\mathfrak{m}_0;\mathfrak{m}_2}_{\mathfrak{m}_1} : \textstyle\prod_{A:\mathcal{U}_{\mathfrak{m}_2}} \Box^{\mathfrak{m}_2}_{\mathfrak{m}_0} A \to \Box^{\mathfrak{m}_1}_{\mathfrak{m}_0} \Box^{\mathfrak{m}_2}_{\mathfrak{m}_1} A$$

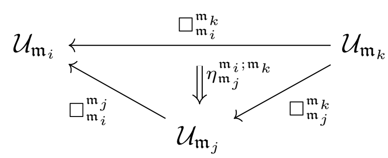
by $\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_2}(A) \equiv \Box_{\mathfrak{m}_0} \eta_{\mathfrak{m}_1}(A)$. This determines a “natural transformation”

$$\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_2} : \Box_{\mathfrak{m}_0}^{\mathfrak{m}_2} \Rightarrow \Box_{\mathfrak{m}_0}^{\mathfrak{m}_1} \Box_{\mathfrak{m}_1}^{\mathfrak{m}_2} : \mathcal{U}_{\mathfrak{m}_2} \to \mathcal{U}_{\mathfrak{m}_0}.$$

Let $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ be LAMs. By naturality, the following diagram commutes.

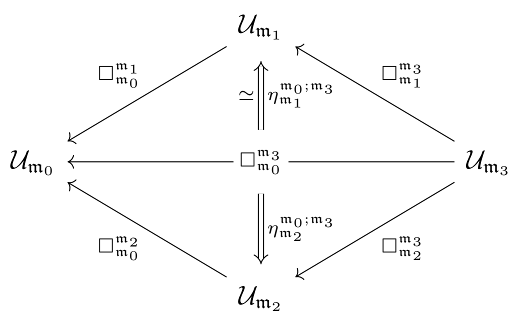


For more than four LAMs, higher coherence laws are also satisfied. Hence, a tuple $(\mathfrak{m}_0, \ldots, \mathfrak{m}_n)$ of LAMs such that $\mathfrak{m}_i \leq {}^{\perp}\mathfrak{m}_j$ for all $i < j$ encodes an n-simplex with vertices $\mathcal{U}_{\mathfrak{m}_i}$, edges $\Box_{\mathfrak{m}_i}^{\mathfrak{m}_j} : \mathcal{U}_{\mathfrak{m}_j} \to \mathcal{U}_{\mathfrak{m}_i}$ for $i < j$, triangles

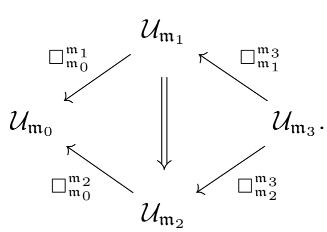


for $i < j < k$, and higher homotopies.

Shapes other than triangles are expressed by postulating invertibility of some of $\eta_{\mathfrak{m}_j}^{\mathfrak{m}_i;\mathfrak{m}_k}$. Let $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ be LAMs and suppose that $\mathfrak{m}_i \leq {}^{\perp}\mathfrak{m}_j$ for all $i < j$. We further assume that $\mathfrak{m}_2 \leq {}^{\perp}\mathfrak{m}_1$ and that $\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_3}$ is invertible. We have



which is equivalent to a diagram of the form



4 Mode sketches

Based on the observation made in Section 3, we introduce *mode sketches* to express shapes of diagrams of subuniverses. We work in HoTT.

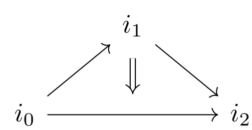
Definition 4.1. A *mode sketch* $\mathfrak{M}$ consists of the following data:

- a decidable finite poset $I_{\mathfrak{M}}$;
- a (decidable) subset $T_{\mathfrak{M}}$ of triangles in $I_{\mathfrak{M}}$. Triangles in $T_{\mathfrak{M}}$ are called *thin*.

Here, by a *decidable* poset we mean a poset whose ordering relation $\leq$ is decidable. Note that the identity relation $i = j$ is equivalent to $(i \leq j) \wedge (j \leq i)$ and thus decidable. The strict ordering $i < j$ defined as $(i \leq j) \wedge (i \neq j)$ is also decidable. By a *triangle* in $I_{\mathfrak{M}}$ we mean an ordered triple $(i_0 < i_1 < i_2)$ of elements of $I_{\mathfrak{M}}$.

Remark 4.2. The definition of mode sketches also makes sense in the metatheory. Every mode sketch $\mathfrak{M}$ in the metatheory can be encoded in type theory since it is finite.

Remark 4.3. We think of a mode sketch $\mathfrak{M}$ as a presentation of a (strict) 2-category $|\mathfrak{M}|$: the 0-cells are the elements of $\mathfrak{M}$; the 1-cells are freely generated by $i \to j$ for every $i < j$ in $\mathfrak{M}$; the 2-cells are generated by



for every $i_0 < i_1 < i_2$ in $\mathfrak{M}$; such a 2-cell is made invertible when $(i_0 < i_1 < i_2)$ is thin; for any chain $i_0 < i_1 < \cdots < i_n$ for $n \geq 3$, the corresponding pasting diagram is made commutative. Alternatively, one can think of a mode sketch as a scaled simplicial set [8] or a stratified simplicial set [19], but the direction of 2-cells is opposite to the standard one.

Remark 4.4. Since $I_{\mathfrak{M}}$ is finite, the underlying category of $|\mathfrak{M}|$ is an *inverse category*, that is, the relation $i \prec j$ defined by "there exists a non-identity morphism $j \to i$" is well-founded. The definition of mode sketches is possibly generalized to allow infinite inverse posets.

Let $\mathfrak{M}$ be a mode sketch and $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ a $\mathfrak{M}$-indexed family of LAMs. We consider the following axioms.

Axiom A. $\mathfrak{m}(i) \leq {}^{\perp}\mathfrak{m}(j)$ for any $j \not\leq i$ in $\mathfrak{M}$.

Axiom B. For any triangle $(i_0 < i_1 < i_2) : T_{\mathfrak{M}}$, the natural transformation $\eta^{\mathfrak{m}(i_0);\mathfrak{m}(i_2)}_{\mathfrak{m}(i_1)} : \Box^{\mathfrak{m}(i_2)}_{\mathfrak{m}(i_0)} \Rightarrow \Box^{\mathfrak{m}(i_1)}_{\mathfrak{m}(i_0)} \Box^{\mathfrak{m}(i_2)}_{\mathfrak{m}(i_1)}$ is invertible.

Axiom C. A type $A : \mathcal{U}$ is contractible whenever $\Box_{\mathfrak{m}(i)} A$ is contractible for every $i : \mathfrak{M}$. (In other words, the top modality is the *canonical join* $\bigvee_{i:\mathfrak{M}} \mathfrak{m}(i)$.)

Remark 4.5. Axiom C is not so important in practical use but excludes models other than our intended models. It is even better to work without Axiom C, because Axioms A and B are stable under restriction to full sub-mode-sketches while Axiom C is not.

Let $\mathfrak{M}$ be a mode sketch and let $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$. Suppose $\mathfrak{m}$ satisfies Axioms A and B. Axiom A implies that $\mathfrak{m}(i) \leq {}^{\perp}\mathfrak{m}(j)$ whenever $i < j$, and thus we have $\Box^{\mathfrak{m}(j)}_{\mathfrak{m}(i)} : \mathcal{U}_{\mathfrak{m}(j)} \to \mathcal{U}_{\mathfrak{m}(i)}$. That is, the family of subuniverses $\lambda i.\mathcal{U}_{\mathfrak{m}(i)}$ is equipped with a contravariant action of 1-cells of $|\mathfrak{M}|$. By the definition of the 2-cells of $|\mathfrak{M}|$ and by Axiom B, $\lambda i.\mathcal{U}_{\mathfrak{m}(i)}$ is equipped with a contravariant action of 2-cells of $|\mathfrak{M}|$. In this way, we think of $\mathfrak{m}$ as a 2-functor from $|\mathfrak{M}|^{\mathrm{op}(1,2)}$, the 2-category obtained by reversing the directions of 1-cells and 2-cells, to the $(\infty, 2)$-category of $(\infty, 1)$-categories.

Example 4.6. Let $\mathfrak{A}[1]$ be

$$0 \longrightarrow 1.$$

Then $\mathfrak{m} : \mathfrak{A}[1] \to \mathsf{LAM}$ consists of two LAMs $\mathfrak{m}(0)$ and $\mathfrak{m}(1)$. Axiom A asserts that $\mathfrak{m}(0) \leq {}^{\perp}\mathfrak{m}(1)$. Axiom B is trivial. Assuming Axioms A and C, we can show that $\mathfrak{m}(1)$ is the *open modality* associated to the proposition $\Box_{\mathfrak{m}(0)} \mathsf{Empty}$ and that $\mathfrak{m}(0)$ is the *closed modality* associated to $\Box_{\mathfrak{m}(0)} \mathsf{Empty}$. This also holds without Axiom C in the sense that, assuming Axiom A, we have another LAM $\mathfrak{n}$ called the *canonical join* of $\mathfrak{m}(0)$ and $\mathfrak{m}(1)$, and $\mathfrak{m}(0)$ and $\mathfrak{m}(1)$ are closed and open, respectively, modalities relative to the subuniverse $\mathcal{U}_{\mathfrak{n}}$. Therefore, Axiom A gives an alternative formulation of *synthetic Tait computability* [14].

Remark 4.7. In Sterling's synthetic Tait computability, a proposition P is postulated and then the open modality $\mathfrak{o}_P$ and the closed modality $\mathfrak{c}_P$ are constructed. This formulation is not stable under embedding of ∞-toposes. Let $\mathcal{X} \subset \mathcal{Y}$ be an embedding of ∞-toposes and P a proposition in the internal language of $\mathcal{X}$. The interpretations of the subuniverses $\mathcal{U}_{\mathfrak{o}_P}$ and $\mathcal{U}_{\mathfrak{c}_P}$ in $\mathcal{Y}$ are different from those in $\mathcal{X}$. In contrast, our formulation is stable under embedding of ∞-toposes. Let $\mathfrak{m} : \mathfrak{A}[1] \to \mathsf{LAM}$ be a family of LAMs in the internal language of $\mathcal{X}$ satisfying Axiom A. $\mathcal{U}_{\mathfrak{m}(i)}$ is interpreted in $\mathcal{X}$ as a subtopos $\mathcal{X}_i \subset \mathcal{X}$, the interpretation of $\mathcal{U}_{\mathfrak{m}(i)}$ in $\mathcal{Y}$ is still $\mathcal{X}_i$.

5 Semantics of mode sketches

We work in the metatheory.

Theorem 5.1. *Let $\mathfrak{M}$ be a mode sketch. We have an equivalence between the following spaces.*

1. *∞-toposes equipped with $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ in their internal languages satisfying Axioms A, B and C.*
2. *$|\mathfrak{M}|$-indexed diagrams of ∞-toposes and lex, accessible functors between them.*

Moreover, the map from 2 to 1 is given by the oplax limit construction.

Example 5.2. Let $\mathfrak{A}[1] = \{0 \to 1\}$ (Example 4.6). A $|\mathfrak{A}[1]|$-indexed diagram of ∞-toposes and lex, accessible functors between them consists of two ∞-toposes $\mathcal{X}_0$ and $\mathcal{X}_1$ and a lex, accessible functors $F : \mathcal{X}_1 \to \mathcal{X}_0$. Its oplax limit is the so-called *Artin gluing* $\mathbf{Gl}(F)$ and defined by the pullback

$$\begin{array}{ccc} \mathbf{Gl}(F) & \longrightarrow & \mathcal{X}_0^{\to} \\ \big\downarrow & \lrcorner & \big\downarrow \mathbf{cod} \\ \mathcal{X}_1 & \xrightarrow[F]{} & \mathcal{X}_0. \end{array}$$

In other words, $\mathbf{Gl}(F)$ is the $(\infty, 1)$-category of triples (A_0, A_1, f) consisting of objects $A_0 \in \mathcal{X}_0$ and $A_1 \in \mathcal{X}_1$ and a map $f : A_0 \to F(A_1)$.

The projections $\mathbf{Gl}(F) \to \mathcal{X}_0$ and $\mathbf{Gl}(F) \to \mathcal{X}_1$ have fully faithful right adjoints $A_0 \mapsto (A_0, \mathbf{1}, !)$ and $A_1 \mapsto (F(A_1), A_1, \mathbf{id})$, respectively, and exhibit $\mathcal{X}_0$ and $\mathcal{X}_1$, respectively, as subtoposes of $\mathbf{Gl}(F)$. Let $\mathfrak{m}(i)$ be the LAM in the internal language of $\mathbf{Gl}(F)$ corresponding to $\mathcal{X}_i$ for $i = 0, 1$. Then $\mathfrak{m}$ satisfies Axioms A, B and C.

Conversely, let $\mathcal{X}$ be an ∞-topos equipped with a $\mathfrak{m} : \mathfrak{A}[1] \to \mathsf{LAM}$ in its internal language satisfying Axioms A, B and C. Let $\mathcal{X}_i$ be the subtoposes of $\mathcal{X}$ corresponding to $\mathfrak{m}(i)$ for $i = 0, 1$. We have a lex, accessible functor $F : \mathcal{X}_1 \to \mathcal{X}_0$ by externalizing $\square^{\mathfrak{m}(1)}_{\mathfrak{m}(0)}$. The *fracture and gluing theorem* [11, Corollary 3.52] gives an equivalence

$$\mathcal{U} \simeq \textstyle\sum_{A_0 : \mathcal{U}_{\mathfrak{m}(0)}} \sum_{A_1 : \mathcal{U}_{\mathfrak{m}(1)}} A_0 \to \square^{\mathfrak{m}(1)}_{\mathfrak{m}(0)} A_1$$

in the internal language of $\mathcal{X}$. Externalizing it, we have an equivalence

$$\mathcal{X} \simeq \mathbf{Gl}(F).$$

Definition 5.3. Let I be an $(\infty, 2)$-category. By an *I-indexed $(\infty, 1)$-category* we mean a family of $(\infty, 1)$-categories $\{\mathcal{C}_i\}_{i \in I}$ equipped with contravariant actions of 1-cells and 2-cells in I.

Construction 5.4. Let I be an $(\infty, 2)$-category and $\mathcal{C}$ an I-indexed $(\infty, 1)$-category. The *oplax limit* $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ is the $(\infty, 1)$-category described as follows. An object x of $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ consists of the following data:

- an object $x_i \in \mathcal{C}_i$ for any object $i \in I$;
- a morphism $x_\alpha : x_i \to x_j \cdot \alpha$ for any morphism $\alpha : i \to j$ in I;
- some coherence data.

A morphism $x \to y$ in $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ is a family of morphisms $x_i \to y_i$ for $i \in I$ equipped with some coherence data.

Proof of Theorem 5.1. Let $\mathcal{X}$ be a $|\mathfrak{M}|$-indexed diagram of ∞-toposes and lex, accessible functors between them. For every $i \in \mathfrak{M}$, the projection

$$\operatorname*{opLaxLim}_{j \in \mathfrak{M}} \mathcal{X}_j \to \mathcal{X}_i$$

exhibits $\mathcal{X}_i$ as a subtopos of $\operatorname{opLaxLim}_{j \in \mathfrak{M}} \mathcal{X}_j$. This determines a function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ in the internal language of $\operatorname{opLaxLim}_{i \in \mathfrak{M}} \mathcal{X}_i$ satisfying Axioms A, B and C.

Conversely, let $\mathcal{X}$ be an ∞-topos equipped with $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ in its internal language satisfying Axioms A, B and C. $\mathcal{U}_{\mathfrak{m}(i)}$'s form a $|\mathfrak{M}|$-indexed diagram of subuniverses. Iterating the fracture and gluing, one can see that $\mathcal{U}$ is the oplax limit of $\mathcal{U}_{\mathfrak{m}(i)}$'s in the internal language. By externalization, we have $\mathcal{X} \simeq \operatorname{opLaxLim}_{i \in \mathfrak{M}} \mathcal{X}_i$ where $\mathcal{X}_i \subset \mathcal{X}$ is the subtopos corresponding to $\mathfrak{m}(i)$. □

A Oplax limits

Theorem A.1. *Let I be an $(\infty, 2)$-category and $\mathcal{X}$ an I-indexed $(\infty, 1)$-category. Suppose:*

- *$\mathcal{X}_i$ is an ∞-topos for any object $i \in I$;*
- *$\mathcal{X}_j \to \mathcal{X}_i$ is lex and accessible for any morphism $\alpha : i \to j$ in I.*

Then $\operatorname{opLaxLim}_{i \in I} \mathcal{X}_i$ is an ∞-topos. Moreover, colimits and finite limits in $\operatorname{opLaxLim}_{i \in I} \mathcal{X}_i$ are computed in $\prod_{i \in I} \mathcal{X}_i$.

Proof. One can verify the case when $I = \{0 \to 1\}$. For a general case, decompose I into a colimit of cells and use the fact that the $(\infty, 1)$-category of ∞-toposes and lex, colimit-preserving functors between them are closed under limits of $(\infty, 1)$-categories [9, Proposition 6.3.2.3]. □

References

[1] M. Anel, G. Biedermann, E. Finster, and A. Joyal. "Left-exact localizations of ∞-topoi I: Higher sheaves". In: *Adv. Math.* 400 (2022), Paper No. 108268, 64. DOI: 10.1016/j.aim.2022.108268.

[2] P. Arndt and K. Kapulkin. "Homotopy-Theoretic Models of Type Theory". In: *Typed Lambda Calculi and Applications: 10th International Conference, TLCA 2011, Novi Sad, Serbia, June 1-3, 2011. Proceedings.* Ed. by L. Ong. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, pp. 45–60. DOI: 10.1007/978-3-642-21691-6_7.

[3] R. Bocquet, A. Kaposi, and C. Sattler. *Relative induction principles for type theories.* 2021. arXiv: 2102.11649v2.

[4] J. D. Christensen, M. Opie, E. Rijke, and L. Scoccola. "Localization in homotopy type theory". In: *High. Struct.* 4.1 (2020), pp. 1–32.

[5] J. D. Christensen and E. Rijke. "Characterizations of modalities and lex modalities". In: *J. Pure Appl. Algebra* 226.3 (2022), Paper No. 106848, 21. DOI: 10.1016/j.jpaa.2021.106848.

[6] D. Gratzer. *Normalization for multimodal type theory.* 2021. arXiv: 2106.01414v1.

[7] D. Gratzer, G. A. Kavvos, A. Nuyts, and L. Birkedal. "Multimodal Dependent Type Theory". In: *Logical Methods in Computer Science* Volume 17, Issue 3 (July 2021). DOI: 10.46298/lmcs-17(3:11)2021.

[8] J. Lurie. *$(\infty, 2)$-Categories and the Goodwillie Calculus I.* 2009. arXiv: 0905.0462v2.

[9] J. Lurie. *Higher Topos Theory.* Vol. 170. Annals of Mathematics Studies. Princeton University Press, 2009. URL: https://www.math.ias.edu/~lurie/papers/HTT.pdf.

[10] H. K. Nguyen and T. Uemura. *∞-type theories.* 2022. arXiv: 2205.00798v1.

[11] E. Rijke, M. Shulman, and B. Spitters. "Modalities in homotopy type theory". In: *Log. Methods Comput. Sci.* 16.1 (2020), Paper No. 2, 79. DOI: 10.23638/LMCS-16(1:2)2020.

[12] M. Shulman. *All $(\infty, 1)$-toposes have strict univalent universes.* 2019. arXiv: 1904.07004v2.

[13] M. Shulman. "Univalence for inverse diagrams and homotopy canonicity". In: *Mathematical Structures in Computer Science* 25.05 (2015), pp. 1203–1277. DOI: 10.1017/s0960129514000565.

[14] J. Sterling. "First Steps in Synthetic Tait Computability. The Objective Metatheory of Cubical Type Theory". PhD Thesis. Carnegie Mellon University, 2021. URL: https://www.jonmsterling.com/pdfs/sterling:2021:thesis.pdf.

[15] J. Sterling and C. Angiuli. "Normalization for Cubical Type Theory". In: *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS).* 2021, pp. 1–15. DOI: 10.1109/LICS52264.2021.9470719.

[16] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* Institute for Advanced Study, 2013. URL: http://homotopytypetheory.org/book/.

[17] T. Uemura. *Normalization for initial space-valued models of type theories.* WG6 kick-off meeting: Syntax and Semantics of Type Theories. May 2022. URL: https://europroofnet.github.io/wg6-kickoff-stockholm/programme#taichi-uemura-abstract.

[18] M. Vergura. *Localization theory in an ∞-topos.* 2019. arXiv: 1907.03836v1.

[19] D. R. B. Verity. "Weak complicial sets. I. Basic homotopy theory". In: *Adv. Math.* 219.4 (2008), pp. 1081–1149. DOI: 10.1016/j.aim.2008.06.003.