Elementary Simplicial Collapses in Cubical Agda

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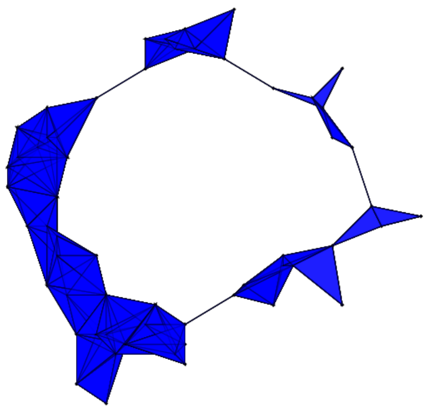
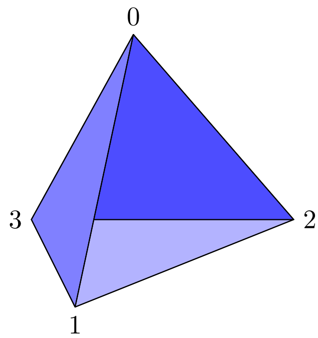
Recent years have seen the rise of applied topology, which aims to harness methods developed in algebraic topology for the analysis of datasets. Given a point cloud, a topological space is constructed which allows to study topological features of the data. In Figure 1a we have depicted points in a metric space and the associated Vietoris-Rips complex [8]. By computing algebraic invariants of the complex such as the fundamental or first homology group we can detect that all points of the dataset lie within a circle. However, computing such invariants will be unnecessarily expensive as the complex contains many superfluous triangulations. To address this problem, discrete Morse theory (DMT) [2] has been developed, which allows to reduce the size of a complex while retaining its topological features. At the heart of DMT is the notion of a simplicial collapse [9], which characterizes under which circumstances a pair of faces of a complex can be removed without changing the homotopy type of the complex.



(a) Vietoris-Rips complex on a dataset



(b) The 3-simplex after a simplicial collapse

Figure 1

Consider Figure 1b, where have we have depicted a 3-simplex after performing an elementary collapse. Intuitively, the 2-dimensional triangle spanned by 2, 1 and 0 has been pushed inside the tetrahedron filler, resulting in a hollowed out tetrahedron with one side missing. Both the 3-simplex and its collapse are contractible and thereby have the same homotopy type.

Discrete Morse theory characterizes which sequences of simplicial collapses are admissible. Intuitively, we need to ensure that no sequence of collapses closes any holes in the complex, which can be done with a certain acyclicity constraint on the set of collapsed pairs. After performing all collapses we are left with a smaller complex, called the Morse complex, which is in general not simplicial, but a cell complex. For instance, if we did a Morse reduction on the complex in Figure 1a with all faces but one part of a collapse, the Morse complex would be a single point with a path to itself, i.e., the circle.

In this talk I will introduce a framework in Cubical Agda [7] in which discrete Morse theory can be studied; and to prove an important first result towards its full formalization: elementary simplicial collapses of standard simplices preserve the homotopy type. In literature on discrete Morse theory, this fact is commonly established by an appeal to topological intuition, see, e.g., [4, p. 22]. In order to formalize this statement in a theorem prover, we need to have available a notion of topological space. Since we are only interested in the homotopy type of the Morse complex and do not need to reason on the level of homeomorphisms, Cubical Agda [7] is up to this task: Cubical Agda takes literally the tenet of homotopy type theory (HoTT) [6], which interprets types of intensional type theory as homotopy types. Cubical Agda offers a primitive notion of space and paths in a space, enabling novel ways of conducting arguments about topological spaces, demonstrated for instance by [3, 1]. We will use this style of reasoning to give a proof of the above statement (with respect to homotopy equivalence), which is very combinatorial in flavour as we reason on the level of cubical sets. Thereby our proof fits in nicely with the general approach of researchers in discrete Morse theory, who prefer combinatorial reasoning over topological arguments where possible [4, 5]. Our formalization is meant as a stepping stone to formalize the main result of [4], who gives an explicit description of the homotopy type of the Morse complex in terms of the localized entrance-path category, which is a certain category associated with cell complexes. This recent result significantly refines traditional accounts of DMT [2], which only give way to recover the homology groups of the Morse complex. To this end, we will give a formalization of simplicial complexes and their entrance-path categories in this paper, localization is left to future work.

The formalization of all results can be accessed at `https://github.com/maxdore/collapse`.

References

[1] Guillaume Brunerie, Axel Ljungström, and Anders Mörtberg. Synthetic cohomology theory in cubical agda. In *Computer Science Logic (CSL'22)*, 2022.

[2] Robin Forman. Morse theory for cell complexes. *Advances in Mathematics*, 134(1):90–145, 1998. `doi:10.1006/aima.1997.1650`.

[3] Anders Mörtberg and Loïc Pujet. Cubical synthetic homotopy theory. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs*, CPP 2020, pages 158–171, New York, NY, USA, 2020. Association for Computing Machinery. URL: `https://doi.org/10.1145/3372885.3373825`, `doi:10.1145/3372885.3373825`.

[4] Vidit Nanda. Discrete morse theory and localization. *Journal of Pure and Applied Algebra*, 223(2):459–488, 2019. doi:10.1016/j.jpaa.2018.04.001.

[5] Vidit Nanda, Dai Tamaki, and Kohei Tanaka. Discrete morse theory and classifying spaces. *Advances in Mathematics*, 340:723–790, 2018. doi:10.1016/J.AIM.2018.10.016.

[6] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.

[7] Andrea Vezzosi, Anders Mörtberg, and Andreas Abel. Cubical Agda: a dependently typed programming language with univalence and higher inductive types. *Proceedings of the ACM on Programming Languages*, 3(ICFP):1–29, 2019. doi:10.1145/3341691.

[8] Leopold Vietoris. Über den höheren zusammenhang kompakter räume und eine klasse von zusammenhangstreuen abbildungen. *Mathematische Annalen*, 97(1):454–472, 1927.

[9] John Henry Constantine Whitehead. Simplicial spaces, nuclei and m-groups. *Proceedings of the London Mathematical Society*, 45(1):243–327, 1939.