A type-theoretic model structure over cubes with one connection presenting spaces

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1 Introduction

It is generally expected that models of homotopy type theory correspond to "elementary $(\infty, 1)$-topoi", for some definition of the latter; it is now known [Shu19] that HoTT at least has an interpretation in any Grothendieck $(\infty, 1)$-topos. It is too much to expect, however, that the strict equalities of type theory appear *literally* in a given $(\infty, 1)$-category. An interpretation of HoTT in an $(\infty, 1)$-category is therefore necessarily filtered through a *presentation* of the category with stricter structure. Said presentation has typically come in the form of a *Quillen model category*.

Not all presentations of an $(\infty, 1)$-category are equal. For example, the $(\infty, 1)$-category of $(\infty, 1)$-*groupoids* (i.e. "spaces") admits numerous well-studied presentations by model categories: on topological spaces, simplicial sets, cubical sets of several kinds, and so on. Grothendieck's theory of *test categories* provides a general class of presheaf model categories presenting spaces [Cis06, Proposition 4.4.28], of which simplicial and various cubical sets are instances. Different presentations may be better-suited for particular constructions; while simplicial sets are the standard choice, cubical sets have their own convenient combinatorial properties, as exploited in e.g. [KV20].

Some varieties of cubical set have recently attracted attention for their ability to *constructively* support model structures interpreting HoTT, whereas the traditional simplicial set interpretation [KL21] requires classical principles [BCP15].[1] Bezem, Coquand, and Huber [BCH13] gave the first constructive interpretation of HoTT in a cubical set category; this was followed by interpretations in other cubical settings [CCHM15; AFH18; ABCHFL21]. While these interpretations draw on model-categorical ideas, they do not formally construct model categories; they are interpretations of HoTT in the literal sense of interpreting its judgments. Associated model structures were formally established by authors beginning with Sattler [Sat17; CMS20; Awo21].

It is not clear in general, however, how to characterize the $(\infty, 1)$-categories these model structures present. Applied to simplicial sets, Sattler's construction does reproduce the standard model structure presenting spaces [Sat17, Corollary 8.4]. However, most cubical set categories considered as models of HoTT do *not* give rise to model structures classically presenting spaces, as was noticed by Buchholtz [Coq+18] and Sattler [Sat18]. This is despite the fact that all of these are presheaves over test categories, thus classically support a test model structure presenting spaces [BM17].

The first constructive interpretation of HoTT supporting a model structure classically presenting spaces was uncovered by Awodey, Cavallo, Coquand, Riehl, and Sattler [Rie20]. This work (currently in preparation) modifies an existing interpretation in *cartesian cubical sets* [Awo18; ABCHFL21; Awo21], imposing a further *equivariance* condition on the lifting property defining fibrations. Here we describe a second example: *cartesian cubical sets with one connection*, or more compactly *disjunctive cubical sets*. We claim that a model structure on this category associated with an interpretation of HoTT supports a Quillen equivalence to the Kan-Quillen model structure $\widehat{\Delta}^{\mathrm{KQ}}$ on simplicial sets, thus presents spaces. We will substantiate the argument sketched below in a forthcoming article.

[1]Henry has since exhibited a constructive model structure on simplicial sets that classically coincides with the standard one [Hen19; GSS19]; its capacity to model HoTT is not completely understood [GH19].

2 Disjunctive cubical sets

Our disjunctive cube category is generated by face, degeneracy, diagonal, permutation, and max-connection maps; in the taxonomy of [BM17], it is $\mathbb{C}_{(\mathrm{wec},\vee)}$. (The choice of $\vee$ over $\wedge$ is arbitrary.) We can give a concise and convenient description as a subcategory of a category of algebras.

Definition 2.1. Write $\vee$-**Alg** for the category of join-semilattices: objects are sets with a commutative, associative, and idempotent binary operator $\vee$, while morphisms are $\vee$-preserving functions.

Definition 2.2. Write $[1]$ for the $\vee$-algebra with underlying set $\{0,1\}$ and $\vee \coloneqq \max$. The *disjunctive cube category* is the full subcategory $\square_\vee \hookrightarrow \vee\text{-}\mathbf{Alg}$ consisting of $[1]^n$ for $n \in \mathbb{N}$.

We write $\mathbf{PSh}(\square_\vee)$ for the category of presheaves on $\square_\vee$ and $\mathbb{I} \in \mathbf{PSh}(\square_\vee)$ for the representable corresponding to $[1] \in \square_\vee$. An interpretation of HoTT and accompanying model structure may be obtained by existing techniques developed for cartesian cubical sets; here we apply [CMS20].

Proposition 2.3. There is a model structure $\widehat{\square}_\vee^{\mathrm{ty}}$ on $\mathbf{PSh}(\square_\vee)$ in which the cofibrations are the monomorphisms, while the fibrations are the maps with the right lifting property against against all pushout products $\delta_\varepsilon \widehat{\times} m$ of an endpoint inclusion $\delta_\varepsilon \colon 1 \to \mathbb{I}$ with a monomorphism $m \colon A \rightarrowtail B$.

Proof. By [CMS20, Theorem 34] with interval $1 \to \mathbb{I}$ and cofibration classifier $\top \colon 1 \to \Omega$. The characterization of fibrations there requires a more general lifting property, but it is equivalent to the above for an interval with a connection and a classifier containing the diagonal $\mathbb{I} \rightarrowtail \mathbb{I} \times \mathbb{I}$. $\square$

Notably, the distinction between "ordinary" and equivariant fibrations disappears in this setting; a connection suffices to show that any fibration supports the *a priori* stronger equivariant lifting.

We construct an adjoint triple relating $\mathbf{PSh}(\square_\vee)$ to simplicial sets with the help of an *idempotent completion* $\blacksquare \colon \square_\vee \hookrightarrow \overline{\square}_\vee$. A closure of $\square_\vee$ under splitting of idempotents [BD86], such a map induces an equivalence of presheaf categories. We give an explicit characterization of $\overline{\square}_\vee$ as the full subcategory of $\vee$-**Alg** consisting of the finite distributive lattices. Conveniently, we also have an embedding $\blacktriangle \colon \Delta \hookrightarrow \overline{\square}_\vee$ of the simplex category, sending the n-simplex to $\{0,\dots,n\}$ with its usual order. By adapting arguments for the cartesian cube category with two connections [Sat19; SW21], we can show that $\blacksquare^*\blacktriangle_! \colon \mathbf{PSh}(\Delta) \to \mathbf{PSh}(\square_\vee)$ is both a left and right Quillen functor. The unit of the adjunction $\blacksquare^*\blacktriangle_! \dashv \blacktriangle^*\blacksquare_*$ is an isomorphism, so its derived unit is valued in weak equivalences.

The counit is more problematic. The obstacle is that $\square_\vee$ is not a (generalized) *Reedy category*, a category in which each object has some *dimension* and which is in some sense generated by certain dimension-lowering *face* and dimension-raising *degeneracy* morphisms [BM11; RV14; Shu15]. Reedy techniques are ubiquitous in homotopy theory; diagrams over a Reedy category can be studied in an "inductive" fashion, iteratively considering indices of increasing dimension. These techniques are essential to the characterization of the equivariant model structure mentioned above. It is the combination of diagonals and a connection which is problematic in our case: the reader familiar with Reedy categories should contemplate the map $(x,y,z) \mapsto (x \vee y, y \vee z, z \vee x) \colon [1]^3 \to [1]^3$.

We observe that while $\square_\vee$ is not Reedy, we do have an inclusion $i \colon \square_\vee \hookrightarrow \vee\text{-}\mathbf{Alg}_{\mathrm{fin}}$ into the Reedy category of finite join-semilattices. Moreover, this embedding satisfies a relativized version of the criteria defining an *Eilenberg-Zilber (EZ)* or *elegant* Reedy category [BM11, Definition 1.1]. Namely, whereas an EZ category $\mathbf{R}$ has pushouts of spans of degeneracy maps preserved by the Yoneda embedding $\mathbf{R} \hookrightarrow \mathbf{PSh}(\mathbf{R})$, the category $\vee\text{-}\mathbf{Alg}_{\mathrm{fin}}$ has pushouts of spans of degeneracy maps preserved by the *nerve* $N_i \colon \vee\text{-}\mathbf{Alg}_{\mathrm{fin}} \to \mathbf{PSh}(\square_\vee)$. We say that $\vee\text{-}\mathbf{Alg}_{\mathrm{fin}}$ is *elegant relative to* i. In a category of presheaves over an EZ category $\mathbf{R}$, any property *saturated by monomorphisms* [Cis19, Definition 1.3.9]—that is, closed under certain colimits—can be checked for all presheaves by showing that it holds for quotients of representables by groups of automorphisms. We show that if $\mathbf{R}$ is instead elegant relative to some $i \colon \mathbf{C} \hookrightarrow \mathbf{R}$, then such a property can be checked by showing that it holds for automorphism quotients of objects in the image of N_i.

The collection of $X \in \mathbf{PSh}(\square_\vee)$ for which the counit of $\blacksquare^*\blacktriangle_! \dashv \blacktriangle^*\blacksquare_*$ is a weak equivalence is saturated by monomorphisms. It is then straightforward to check that the counit is a weak equivalence at the generators specified above, completing the construction of our Quillen equivalence.

Theorem 2.4. $\blacksquare^*\blacktriangle_! \dashv \blacktriangle^*\blacksquare_*$ defines a Quillen equivalence between $\widehat{\Delta}^{\mathrm{KQ}}$ and $\widehat{\square}_\vee^{\mathrm{ty}}$.

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