Towards Normalization of Simplicial Type Theory via Synthetic Tait Computability

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1 Introduction

Riehl and Shulman [12] introduced a simplicial extension of (homotopy) type theory to reason synthetically about $(\infty, 1)$-categories. Indeed, the semantics of this theory matches up with established results from homotopy theory, *i.e.*, the synthetic $(\infty, 1)$-categories in the theory correspond externally to internal $(\infty, 1)$-categories in an arbitrary given ∞-topos (implemented as complete Segal objects), cf. [10, 14, 23]. However, certain meta-theoretic properties of this simplicial type theory (STT), such as canonicity and normalization, have not been investigated yet.

In this talk, we explain our progress towards establishing a normalization result for STT (without any axioms such as function extensionality) using the framework of *synthetic Tait computability (STC)*. The latter is due to Sterling and Harper [17] and has been used by Sterling–Angiuli [16] and Sterling [15] to establish meta-theoretical results for cubical type theory (CTT) such as canonicity (cf. [13]) and normalization, moreover in a syntax-invariant manner.

Contribution In the talk, we'll at first briefly review the method of normalization by evaluation (NBE) in a more classical setting (cf. *e.g.* [6]). We then show how to establish an analogous version for the case of STT, which is currently under development.

In particular, along the we explain how to construct a computability topos for simplicial type theory, analogously to the work [16, 15] for cubical type theory.

2 Simplicial Type Theory

Simplicial type theory (STT) has been introduced by Riehl–Shulman [12] as an extension of Martin–Löf type theory by a strict notion of shape inclusions and *extension types* capturing strict extensions of terms along such inclusions. This in particular makes it possible to define *e.g.* hom-types and a synthetic notion of $(\infty, 1)$-category using internal versions of the Segal condition and Rezk completeness.

Building on fundamental parts of synthetic higher category theory developed in [12] there has been work on synthetic $(\infty, 1)$-categories in the simplicial setting [11, 4, 23, 9, 24, 22] and in the bicubical setting [21]. Kudasov is working on a prototype proof assistant supporting STT [8].

3 Simplicial Synthetic Tait Computability

Presentation as a fibered signature In his recent PhD thesis [15], Sterling develops a logical framework to define a variety of type theories. The idea is to present a type theory by a *signature*, which specifies abstractly the potential judgments to be formed. The actual admissible contexts of the type theory are organized into a *category of atomic contexts* which arises as a submodel of the syntactic model.

Simplicial STC and computability topos We adapt the axiomatization from [16] to the setting of simplicial rather than cubical type theory. One notable difference is that the simplicial interval $\mathbb{2}$ is *not* tiny, *i.e.*, exponentiation $(-)^{\mathbb{2}} : \mathcal{E} \to \mathcal{E}$ does not have a right adjoint.

Namely, we devise an axiomatization of an ambient category $\mathcal{E}$ whose internal language supports an appropriate notion of computability structure. This can be instantiated by a topos pushout (actually a gluing presheaf topos) of an open (syntactic) with a closed (semantic) subtopos, using an appropriate simplicial *figure shape*, as in [16, 15].

Towards normalization of STT Along the lines of [16, 15] we can then carry out a version of normalization by evaluation (NBE), based on an analogous notion of *stabilized neutrals* to capture the (more general) case of extension types.

In the classical picture of NBE, after Tait [20], one constructs for a type A a chain of maps (*reflection* and *reification*)

$$\mathrm{ne}(A) \to [\![A]\!] \to \mathrm{nf}(A)$$

from the *neutral terms* of type A to the (computational) semantics of terms of type A, and from there to the *normal forms* of type A (cf. also [5, 2, 19, 6, 1, 3, 18, 7]).

For CTT, it was the insight of Sterling–Angiuli that the conditions for a term to be neutral could not be formulated reasonably. However, they were able to instead capture the conditions for a term to *cease* to be neutral. Informally, if p denotes a path term and i an interval variable, the neutral path application term $p(i)$ would cease to be neutral in case $i \equiv 0$ or $i \equiv 1$ (since this would force another computation). This gives rise to a modified version of Tait's method, called *stabilized* Tait yoga.

We are re-using this idea for the case of STT where this *frontier of instability* for application $\tau(t)$ of a section $\tau : \left\langle \prod_{\Psi} A\middle|^{\Phi}_{\sigma} \right\rangle$ to a tope variable $t : \Psi$ depends on t satisfing the condition $\varphi(t)$ defining the distinguished subshape Φ.

With those modifications, our current progress indicates that the methods by Sterling–Angiuli and Sterling to prove normalization carry over well to the simplicial setting. We will report on our progress in this program.

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