The Compatibility of MF with HoTT

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The Minimalist Foundation (MF) is a two-level foundation for constructive mathematics, that was first conceived in [5] and then fully developed in [2]. It consists of an intensional level, called mTT, an extensional level, called emTT, and an interpretation of the latter into the first through a quotient model construction (which was related to a free categorical construction in [6]). Both levels of MF are extensions of versions of Martin-Löf's type theory with a primitive notion of proposition. One of the main ideas behind the introduction of MF was to design a theory that could be regarded as the common core between the most relevant foundational frameworks for mathematics. In particular, for any of these other settings, an appropriate level of MF should be interpretable in it by means of a 'compatible translation', namely a translation preserving the meaning of logical and set-theoretical constructors so that proofs of mathematical theorems in one theory are understood as proofs of mathematical theorems in the target theory with the same meaning.

So far, no level of MF has been shown to be compatible with any of the Univalent Foundations in the literature, including the first version HoTT presented in the standard reference [9]. Here we show that both levels of MF are compatible with HoTT. In particular, we interpret MF-types as homotopy sets and MF-propositions as h-propositions. Therefore, the distinction between propositions and set-constructors that we have in the syntax of MF is preserved by the translation. This should be contrasted with what happens in the intensional version of Martin-Löf Type Theory: there we can interpret only the intensional level of MF by identifying propositions with sets.

The new machinery available within the context of HoTT, in particular the Univalence Axiom and higher inductive quotients, are essential to achieve these goals.

The main difficulty encountered in this work concerns the interpretation of the extensional level emTT of MF. Since emTT includes Martin-Löf's extensional propositional equality in the style of [7], there is no straightforward way of interpreting it in HoTT.

We managed to solve this issue by employing a technique already used in [2] to interpret emTT over the intensional level of MF: emTT-types and terms are interpreted as HoTT-types and terms up to a special class of isomorphisms, called *canonical* as in [2], by providing a kind of realizability interpretation in the spirit of the interpretation of *true judgements* in Martin-Löf's type theory described in [7, 8]. We introduce the category $\mathsf{Set}_{\mathcal{U}_1}/{\cong_c}$ of h-sets equated under canonical isomorphisms and then we define an interpretation of emTT-judgements into it. In particular emTT-type and term judgements are interpreted as HoTT-type and term judgements up to canonical isomorphisms. Furthermore, the emTT-definitional equality $A = B\ type\ [\Gamma]$ of two emTT-types $A\ type\ [\Gamma]$ and $B\ type\ [\Gamma]$ is interpreted as the existence of a canonical isomorphism that connects the HoTT-type representatives interpreting the emTT-types $A\ \ type\ [\Gamma]$ and $B\ type\ [\Gamma]$, which turn out to be propositionally equal in HoTT thanks to Univalence. In turn, this interpretation is based on another auxiliary partial (multi-function) interpretation of emTT-raw syntax into HoTT-raw syntax, which makes use of canonical isomorphisms.

It must be stressed that the resulting interpretation of emTT into HoTT is simpler than that of emTT within mTT in [2], since we can avoid any quotient model construction thanks to higher inductive (effective) quotients and Univalence. Moreover, the interpretation of emTT into mTT does not show the compatibility of emTT with mTT because of the lack of Univalence and effective quotients in mTT.

We conclude by observing that it does not appear possible to identify "compatible" subsystems of HoTT corresponding to each level of MF: in HoTT the interpretation of the existential quantifier allows to derive both the axiom of unique choice and the rule of unique choice as happens in the internal logic of a topos as in [1], while in each level of MF these principles are not generally valid thanks to results in [3, 4, 6], since the existential quantifier in MF is defined in a primitive way.

References

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