Models of homotopy type theory from the Yoneda embedding

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The connection between logics and categories is important and has been studied, notably using a fibrational approach, which has resulted in categorical notions such as that of comprehension categories to connect the syntax of (dependent) type theory and the world of categories. From a semantical point-of-view, there is a well-known correspondence between (the models of) Martin-Löf type theory and locally cartesian closed categories.

Motivated by this 1-categorical result, the natural question that arose about the link between $(\infty, 1)$-categories and homotopy type theory has recently found some answers extending the previous logical correspondence to quasicategories, through the simplicial model for homotopy type theory ([KL12]) and more generally the so-called internal language of $(\infty, 1)$-topoi (the higher version of Grothendieck topoi) ([Shu19]).

Models of $(\infty, 1)$-categories such as quasicategories come with an inbuilt notion of homotopy, which is compatible with all the structure corresponding to some usual structure of 1-categories (for instance limits). However, while this is very convenient to talk about homotopy coherent properties of quasicategories, there happens to be a mismatch with the usual approach to type theory which involves "strict" features, even when it comes to describing the syntax of homotopy type theory.

Unsurprisingly, interpreting type theory in a higher topos involves a rigidification step to make sense of the expected strict rules of type theory. This can account for the fact that the current results only apply to $(\infty, 1)$-categories which can be presented by a model category: such a model category works as rigidified substitute of, let say, a quasicategory, and presents the very same data in a 1-categorical way (which is more directly suitable to interpret type theory).

One way we may think about the problem is that of coherence. Morphisms (which are central to a categorical interpretation of type theory) in an $(\infty, 1)$-category exist, but there is no internal way to reason about them strictly as everything works up to homotopy: for instance, composition of morphisms is "only" defined up to homotopy. Therefore, if we want to deal with morphisms and their composites, we need to be able to compose them (so that associativity and unitality are strict) in a coherent fashion. This coherence problem is reminiscent of the splitting procedure for a Grothendieck fibration introduced by Bénabou ([Bén80]), whose importance for solving the usual coherence problem arising from pullbacks so as to model substitution in type theory has been noticed, see [Hof94].

Following Bénabou's idea, we may want to rely on a categorical device with inbuilt composition to replace morphisms (and objects). But this is precisely the role played by the slice categories, or, equivalently, the representable presheaves! Indeed, the Yoneda embedding allows us to see any small category $\mathbf{C}$ as a full subcategory of $\mathbf{Set}^{\mathbf{C}^{op}}$ (or of $\mathbf{Cat}_{/\mathbf{C}}$ through the Grothendieck construction).

Our main result consists in providing models of Martin-Löf type theory for a more general class of quasicategories than in the current literature. Having shown that structure such as limits, dependent products and object classifiers in a quasicategory can be carried over to a corresponding simplicial category to model the associated type-theoretic notions, we most notably observe that elementary higher topoi (that is, locally cartesian closed quasicategories with finite limits and colimits, a subobject classifier and enough universes, see [Ras18]) provide models for homotopy type theory.

Explicitly, starting from a quasicategory $\mathcal{C}$, we consider the simplicial category $\overline{\mathcal{C}}$ given as the full subcategory of $\mathbf{SSet}^{\mathfrak{C}(\mathcal{C})^{op}}$ spanned by (injectively fibrant replacements of) the representable simplicial presheaves. We then replace $\overline{\mathcal{C}}$ by the smallest full subcategory of $\mathbf{SSet}^{\mathfrak{C}(\mathcal{C})^{op}}$ containing $\overline{\mathcal{C}}$ stable under taking pullbacks and dependent products along fibrations, as well as forming the factorizations given by the injective model strucutre. It can be shown that, if $\mathcal{C}$ is locally cartesian closed, the strict structure added previously was already present up to equivalence in the sense that all the objects of $\overline{\mathcal{C}}$, which now constitutes a simplicial π-tribe (as introduced in [Joy17]), are still equivalent to representable presheaves. This ensures that the simplicial nerve of $\overline{\mathcal{C}}$ is still equivalent (as a quasicategory) to $\mathcal{C}$. We then prove that univalent universes existing in $\mathcal{C}$ carry over to $\overline{\mathcal{C}}$ in a similar fashion relying on techniques introduced in [Shu19], and make use of initial structures inside $\mathcal{C}$ to provide (higher) inductive types for the underying model of type theory (e.g. through a comprehension category structure) following [LS20].

This also allows us to extend a partial comparison between (some classes of) models of type theory and (some classes of) $(\infty, 1)$-categories provided in [KS19], as conjectured in [KL18].

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