THE INTERNAL AB AXIOMS

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We define and study the Grothendieck AB axioms [4] for abelian (univalent) categories in Homotopy Type Theory. Our main result is that categories of modules over a ring satisfy the internal versions of axioms AB3 through AB5. We deduce this by proving more generally that AB5 (which includes AB3, but not AB4) implies AB4, for any abelian category in HoTT. These facts are all standard in ordinary homological algebra, but become more subtle in a constructive setting such as ours. This work is part of an ongoing effort to develop homological algebra in HoTT, with the goal of furthering the synthetic development of homotopy theory.

Background. In his seminal *Tôhoku paper* [4], Grothendieck developed homological algebra in the abstract setting of abelian categories so as to unify the treatment of sheaf cohomology over a space with the theory of derived functors on module categories. This made the tools of homological algebra available in the setting of sheaves, as was needed by algebraic geometers and others. These tools have since become standard in many branches of mathematics.

A tenet of this development is to work with abelian categories $\mathscr{A}$ satisfying additional properties called the **AB axioms**.[1] We discuss three:

(AB3) for a small set X, the coproduct $\bigoplus_X A$ of any family $A\colon X \to \mathscr{A}$ of objects in $\mathscr{A}$ exists;
(AB4) AB3 holds, and the coproduct functor $\bigoplus_X\colon \mathscr{A}^X \to \mathscr{A}$ is exact for any small set X;
(AB5) AB3 holds, and filtered colimits in $\mathscr{A}$ are exact.

Abelian categories that satisfy AB3–5 and have a *generator* are nowadays called **Grothendieck categories**. Examples are modules over a ring as well as sheaves of $\mathscr{O}_X$-modules.

However there are interesting abelian categories in the wild that fail to be Grothendieck. For example categories of constructible sheaves, or modules over a ring in an elementary topos. Neither of these examples have arbitrary coproducts (AB3) in general. For the latter, simply consider a non-cocomplete elementary topos such as finite sets.

Apart from their intrinsic interest, tools developed in *internal* homological algebra apply in the generality of both of the given examples via the internal language of a topos. Such tools have been developed by Harting [5] and more recently by Blechschmidt [2, 1]. In addition, constructive accounts—of which there are too many to mention—also feed into the internal development by translating them into the internal language.

In HoTT, the basics of abelian precategories have previously been formalised in the UniMath library [8].

Results. The AB axioms as phrased above can be interpreted directly into HoTT, with filtered categories defined in the obvious way. A stepping stone to our main results is the following, whose proof is a careful translation and slight generalisation of [3, Theorem 2.13.4].

Proposition. *Filtered colimits of sets commute with finitely generated limits.*

This result is usually stated for finite limits, which is also all we need below. We omit the definition of 'finitely generated' here, but an interesting corollary is that taking fixed points of G-sets for a f.g. group G commutes with filtered colimits.

There are various ways of seeing that module categories are cocomplete and therefore satisfy AB3. Since filtered colimits of modules can be computed on the underlying sets, the proposition implies that module categories satisfy AB5 as well. It remains to discuss AB4.

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[1] Our account of the AB axioms is borrowed from the Stacks project (079A) which differs slightly from [4].

Classically, we often *define* the coproduct $\bigoplus_X A$ of a family of modules to be the finitely supported elements inside the product $\Pi_X A$. It then follows that $\bigoplus_X$ is exact. However, in our setting there may be no nontrivial maps $\bigoplus_X A \to \Pi_X A$. Nevertheless, we show the following:

Theorem. *Let $\mathscr{A}$ be an abelian category satisfying AB5, and let X be a small type. The functor* $\mathrm{colim}_X : \mathscr{A}^X \to \mathscr{A}$ *preserves products. If X is a set, then the coproduct $\bigoplus_X \equiv \mathrm{colim}_X$ is exact. In particular, AB5 implies AB4.*

Semantically, colim_X yields a left adjoint to base change of modules over X. The theorem has applications to the semantics of injectivity, briefly discussed below.

In ordinary homological algebra one proves that AB5 implies AB4 by replacing a set-indexed family $A\colon X \to \mathscr{A}$ by a diagram on the finite subsets of X, which is filtered. Constructively, however, neither the finite ordered subsets nor the Bishop-finite subsets of X form filtered categories. Roughly, the reason is that these are not closed under unions. Instead we work with ordered finite *sub-multisets*, which do form a filtered category. This approach is inspired by Harting's construction of the *internal coproduct* [5] of abelian groups in an elementary topos.

Grothendieck categories in HoTT are abelian categories which satisfy AB3 through AB5 and have a specified generator. Since module categories are cocomplete and satisfy AB5, we deduce:

Corollary. *Let R be a ring. The category of R-modules is Grothendieck.*

On the previous page, we saw that a category of modules in an elementary topos $\mathscr{E}$ may not *externally* be Grothendieck. In particular, if $\mathscr{E}$ is not cocomplete then one should not expect AB3 to hold. However, any elementary topos is *internally* cocomplete, and for this reason a category of modules in $\mathscr{E}$ does satisfy AB3 internally. Thus working internally repairs, in some sense, certain defects of the external approach.

These results are being formalised in the Coq HoTT library [7].

Applications. Some theorems about Grothendieck categories in ordinary homological algebra, such as the fact that they have enough injectives, are *nonconstructive*. Even so, we hope that parts of the ordinary theory will go through, or have important analogues, constructively.

Exactness of coproducts (AB4) in the elementary setting [5] was key to understanding the relation between *external*, *local*, and *internal* notions of injectivity [6, 2]. We were lead to study AB4 for module categories in order to understand the semantics of injectivity in HoTT, which in turn has applications to the semantics of Ext groups. The latter we have studied with Dan Christensen with the goal of proving a universal coefficient theorem.

References

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