Groupoidal Realizability: Formalizing the Topological BHK Interpretation

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The BHK interpretation is an informal, constructive explanation of logical formulas: it prescribes what counts as evidence for (or a proof of, or a witness to) a given formula. Realizability interpretations are said to formalize the BHK interpretation, realizers playing the role of evidence.

The first realizability interpretation, Kleene's *number realizability* [Kle45] formally bridges the theory of Heyting arithmetic and that of computable functions. Realizability categories constitute universes for computable mathematics—computable with respect to the realizability structure that the category is built upon. Hyland's *effective topos* **Eff** [Hyl82], probably the most famous example of a realizability category, is constructed from *Kleene's first algebra* $\mathscr{K}_1$, and its internal logic extends Kleene's number realizability. More generally, one most often sees realizability categories built over partial combinatory algebras (PCAs, models of untyped computation, of which $\mathscr{K}_1$ is an example); realizers in this setting are computational or algorithmic in nature. That being said, one may perform realizability constructions over different structures, such as *typed* PCAs [Lon99], algebraic lattices [BBS04] and even categories [Bir00, RR01].

Realizability toposes such as **Eff** are, of course, able to model dependent type theory. But for this purpose one can work with subcategories of *assemblies* (over a PCA), which have an elementary presentation. Further subcategories of *modest sets* (wherein elements are determined uniquely by realizers)—equivalently, partial equivalence relations (PERs)—give rise to impredicative universes (closed under "large" products, cf. [Hyl88]). Realizability toposes can actually be constructed as exact completions of categories of assemblies, which themselves arise as regular completions of categories of *partitioned* assemblies [CFS88, Men00].

Recent work on realizabilty in the context of intensional type theory (ITT) or homotopy type theory (HoTT) has been motivated by the search for impredicative and univalent universes of higher homotopy types. The most well studied approach is that of cubical assemblies. In particular, Uemura [Uem19] constructs a model of HoTT in cubical objects valued in assemblies over $\mathscr{K}_1$ that contains an impredicative and univalent universe refuting propositional resizing. Interestingly, Uemura states that realizers in this model "seem to play no role in its internal cubical type theory" (p. 16).

Aside from cubical assemblies, Hofstra and Warren [HW13] equip the syntax of 1-truncated ITT with a notion of realizability , which allows them to show that the syntactic groupoid associated to the type theory generated by a graph has the same homotopy type as the free groupoid on this graph. Moreover, van den Berg [van20] exhibits the effective topos as a path category in which there is an impredicative and univalent universe of propositions satisfying propositional resizing (this models a type theory in which all computation rules are propositional).

We seek to develop realizability models of ITT and (book-style) HoTT in which realizers themselves carry higher-dimensional structure. This is in contrast to the cubical assemblies approaches, where realizers come from the same kind of thing as in traditional, set-based realizability. We take the assemblies approach to realizability categories, equipping the

Hofmann-Streicher groupoid model [HS98] with a notion of realizability analogous to how traditional categories of assemblies do so for the set model.

Realizers derive from a "realizer category" $\mathbb{C}$ containing an interval I *qua* co-groupoid. The interval facilitates a notion of homotopy internal to $\mathbb{C}$ as well as a fundamental groupoid construction $\Pi = (-)^I : \mathbb{C} \to \mathbf{Gpd}$. Thus in a "groupoidal assembly", objects of the underlying groupoid are realized by points in some fundamental groupoid and isomorphisms in the underlying groupoid are realized by paths in the fundamental groupoid. This may be understood as formalizing an extension/modification of the BHK interpretation whereby evidence for an identification of two objects is a path between them (as explained in [Uni13]). Functors between groupoidal assemblies are realized by maps in $\mathbb{C}$, with natural transformations realized by homotopies in $\mathbb{C}$. An instructive example of a realizer category is $\mathbf{Ho}(\mathbf{Top^2})$, the category of space-subspace pairs and homotopy classes of continuous maps (we must quotient in order that $I = [0,1]$ satisfies the axioms of an interval on the nose)—which is not a cartesian closed category, let alone locally cartesian closed. We obtain a theory wherein realizers play no role in the higher-dimensional structure of the model by choosing a discrete interval.

One approach to model identity types is to require the realizability relation to behave like a Grothendieck fibration. This means positing extra structure on the realizer category in order to model function spaces. It is possible to construct an impredicative universe of "modest groupoids" in this setting, via the relationship with generalized congruences [BBP99]—these are gadgets by which one can take the quotient of a category, identifying objects as well as morphisms, and play the role traditionally played by PERs. In order to do this, we first posit a universal object $U \in \mathbb{C}$ to make the typed notion of realizability provided by $\mathbb{C}$ into an untyped one [Bir00, LS02].

However, the extra structure required to reconcile identity types and function spaces prohibits our main would-be example of an untyped realizer category from being a genuine example. The category in question is **SCC**, that of *Scott continuous categories* [Ad7], which are a categorification of Scott domains [Sco82]. Thus we will finish by describing work in progress on an approach to groupoidal realizability via partitioned assemblies that aims to admit **SCC** as an example of a realizer category.

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