Towards computing cohomology of dependent abelian groups

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Modern mathematics is permeated with cohomology theories – lot's of problems in geometry, number theory and algebra can be asked in terms of cohomology groups or rings. In algebraic topology, cohomology with constant coefficients in an abelian group plays an important role. There has been a lot of work along these lines in homotopy type theory, see e.g. [Cav15], [Hou17] and [BH20].
There, the k-th cohomology group of a type X with coefficients in an abelian group A is defined as

$$H^k(X, A) :\equiv \|X \to B^k A\|_0$$

i.e. the 0-truncation of the type of maps into the k-th Eilenberg-MacLane space $B^k A$. The latter were constructed in HoTT in [LF14] and are a $(k-1)$-connected, pointed types, such that the k-th loopspace is A:

$$A \cong \Omega^k(B^k A)$$

Especially with the possibility of interpreting HoTT in general $(\infty, 1)$-toposes, it can be very interesting to consider the generalization, where the coefficient group A depends on X. So if A_x is an abelian group for $x : X$, we can adapt the definition, using dependent function types:

$$H^k(X, A) :\equiv \|(x : X) \to B^k(A_x)\|_0 \equiv \left\| \prod_{x:X} B^k(A_x) \right\|_0$$

This definition includes, via interpretation in toposes, both cohomology of sheaves of abelian groups and group-cohomology.
It is not hard to show basic functoriality properties of this construction, as well as that short exact sequences of coefficients, induce long exact sequences of cohomology groups. Some of this was done in [van18][Section 5.4]. What is also not hard to do and not yet to be found in the literature, is to show, that any pushout square induces a Mayer-Vietoris sequence.
That abstract theorems like this, are not hard to show, is some advantage one would also expect when working in higher categories or toposes, as one can see in the introduction of [Lur09] or the nLab entry on applications of higher category theory.
However, due to the expected lack of injective resolutions, the main classical computational gadget, it is quite unclear, how one could possibly connect this definition of cohomology to more computational definitions, like e.g. Čhech-Cohomology or concrete descritpions of group-cohomology given by projective resolutions.

We will present work in progress on an attempt of this, using that this cohomology theory should satisfy the universal property of a derived functor or universal ∂-functor. However, we only have a proof, that cohomology up to degree one satisfies this universal property. The proof uses what we call local resolutions, which is a technique already employed by [Buc60]. Our resolution is based on the idea used in [Wel18][Definition 4.9, Definition 4.11] to construct a canonical trivializing cover for a fibre bundle.
For a degree one cocycle $\chi : (x : X) \to B^1(A_x)$, this local resolution amounts to an injective map for any $x : X$ of type:

$$\iota_x : A_x \to ((\chi_x = *) \to A_x)$$

on coefficients, given by $\iota_x(a) :\equiv (\_ \mapsto a)$.
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References

[BH20] Ulrik Buchholtz and Kuen-Bang Hou. „Cellular Cohomology in Homotopy Type Theory“. In: *Logical Methods in Computer Science* Volume 16, Issue 2 (June 2020). DOI: 10.23638/LMCS-16(2:7)2020. URL: https://lmcs.episciences.org/6518.

[Buc60] David A. Buchsbaum. „Satellites and Universal Functors“. In: *Annals of Mathematics* 71.2 (1960), pp. 199–209.

[Cav15] Evan Cavallo. „Synthetic Cohomology in Homotopy Type Theory“. 2015. URL: https://www.cs.cmu.edu/~rwh/theses/cavallo-msc.pdf.

[Hou17] Kuen-Bang Hou (Favonia). „Higher-Dimensional Types in the Mechanization of Homotopy Theory“. PhD thesis. [2017]. URL: http://www.math.ias.edu/~favonia/files/thesis.pdf.

[LF14] Dan Licata and Eric Finster. „Eilenberg-MacLane Spaces in Homotopy Type Theory“. In: *Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. CSL-LICS ’14. Vienna, Austria: ACM, 2014, 66:1–66:9. ISBN: 978-1-4503-2886-9. DOI: 10.1145/2603088.2603153. URL: http://doi.acm.org/10.1145/2603088.2603153.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009. ISBN: 9780691140490.

[van18] Floris van Doorn. „On the Formalization of Higher Inductive Types and Synthetic Homotopy Theory“. PhD thesis. 2018. URL: https://florisvandoorn.com/papers/dissertation.pdf.

[Wel18] Felix Wellen. „Cartan Geometry in Modal Homotopy Type Theory“. In: *arXiv e-prints*, arXiv:1806.05966 (June 2018), arXiv:1806.05966. arXiv: 1806.05966 [math.DG].