The Nielsen-Schreier Theorem in Homotopy Type Theory

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June 24, 2021

The Nielsen-Schreier theorem states that subgroups of free groups are themselves free groups. Although the statement of the theorem is simple, it is quite tricky to give a direct proof. The original proofs by Nielsen [6] and Schreier [7] were quite long and unintuitive. However, later on more intuitive proofs were developed using ideas from algebraic topology such as those of Chevalley and Herbrand [3][1] and Baer and Levi [1]. This made the Nielsen-Schreier theorem an ideal candidate for formalisation in HoTT, since we can take ideas from the proof in algebraic topology and implement them in a very short and direct manner. I will talk about my proof of Nielsen-Schreier appearing in [8] together with an accompanying formalisation in the Agda proof assistant of the finite index case.[2]

The idea of the proof is essentially as follows. In HoTT we can define groups as pointed, connected, 1-truncated types [2]. Under this definition, we can implement free groups as the higher inductive type of a (1-truncated) wedge of copies of the circle [5], and we can understand subgroups as covering spaces [4]. We show that every covering space of a wedge of circles is the coequalizer of a graph, using a special case of the flattening lemma. We then show that under suitable conditions every coequalizer of a graph is equivalent to a wedge of circles. We in fact consider two different sets of conditions where this holds. For the first we restrict to the special case of finite index subgroups, but give an entirely constructive proof, which has been formalised in Agda. For the second, we obtain the full Nielsen-Schreier theorem, but need the additional (non constructive) assumption of the axiom of choice. The paper includes a proof that the axiom of choice is strictly necessary, even with the law of excluded middle, by giving an example of a boolean ∞-topos where the Nielsen-Schreier theorem is false, the "Schanuel ∞-topos".

I will talk in particular about the construction of spanning trees. This is sometimes overlooked in more abstract accounts of the Nielsen-Schreier theorem, but forms a key step in the proof that the coequalizer of a graph is a wedge of circles, and works surprisingly well in HoTT. Usually in graph theory, we define a path as a finite sequence of edges. However, in HoTT we can use a simpler

[1] The Chevalley-Herbrand proof in fact used Riemann surfaces.

[2] Available at `https://github.com/awswan/nielsenschreier-hott`.

definition, based on a topological understanding, that is easier to deal with in formalisations. We define a path to be an element of the identity type of the coequalizer of the graph. In particular we can define a graph to be a *tree* when its coequalizer is contractible. The Agda formalisation works entirely with these simpler definitions, avoiding the usual, more elaborate definitions in terms of sequences of edges.

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