EXT GROUPS IN HOMOTOPY TYPE THEORY

J. DANIEL CHRISTENSEN AND JARL G. TAXERÅS FLATEN

We study Ext groups in homotopy type theory and their semantic counterparts in a model [7]. Our setting is that of the HoTT Book [10], in particular we assume a hierarchy of univalent universes. The results are being formalized in the Coq HoTT library [9].

It is conjectured that all internal homotopy groups of spheres in an ∞-topos are (internally) isomorphic to the classical ones. This indicates that one ought to be able to prove classical facts about $\pi_k(S^n)$ in homotopy type theory, and indeed this has already been done for some k and n. An obstruction to internalising classical proofs is that they sometimes rely on nonconstructive results from homological algebra. One specific example is $\pi_5(S^3)$, which has not yet been computed in homotopy type theory, and whose classical proof uses the universal coefficient theorem (which is nonconstructive). It therefore seems essential to understand homological algebra in homotopy type theory, and developing theory about Ext is part of this endeavor.

Given abelian groups[1] A and B in homotopy type theory, we define **the type $\mathcal{E}xt^1(B,A)$ of abelian extensions of B by A** as the Σ-type over all short exact sequences:

$$0 \to A \xrightarrow{i} E \xrightarrow{p} B \to 0$$

This is analogous to the classical Yoneda Ext [4], available in any abelian category. The **set of extensions of B by A** is defined as the set-truncation: $\mathrm{Ext}^1(B,A) :\equiv \|\mathcal{E}xt^1(B,A)\|_0$, and this is an abelian group under the Baer sum with 0 being the split extension $A \to A \oplus B \to B$.

Classically, the category of abelian groups has homological dimension 1, meaning higher Ext groups vanish. This is an immediate consequence of (i) the Nielsen–Schreier theorem (subgroups of free groups are free); (ii) that all abelian groups A admit an epimorphism $F \to A$ from a free group F; and (iii) that free abelian groups F are projective (i.e. $\mathrm{Ext}^1(F,-) = 0$). However (i) and (iii) fail to hold for abelian groups in homotopy type theory. The Nielsen–Schreier theorem has been studied by Swan [8] in this context, and (iii) is well-known to fail in a constructive setting. This justifies studying higher Ext of abelian groups in homotopy type theory, and in this talk we will present a concrete example of a nontrivial Ext^2 in a model. Our work shows moreover that such examples are already known, which we explain below.

The groups $\mathrm{Ext}^n(B,A)$ are defined recursively as the profunctor tensor product:

$$\mathrm{Ext}^{n+1}(B,A) :\equiv \|\mathrm{Ext}^1(B,-) \times_{\mathrm{Set}} \mathrm{Ext}^n(-,A)\|_0$$

This description is a theorem classically, which follows from results in [2]. The resulting induction principle lets us internalise the arguments in [4] to construct the long exact sequence:

$$\cdots \to \mathrm{Ext}^n(A,G) \to \mathrm{Ext}^{n+1}(C,G) \to \mathrm{Ext}^{n+1}(B,G) \to \mathrm{Ext}^{n+1}(A,G) \to \cdots$$

associated to a short exact sequence $A \to B \to C$ and an abelian group G.

In order to extract information from the long exact sequence, we must be able to compute Ext groups by other means. This is often done classically via projective (or injective) resolutions. We identify the abelian groups P such that $\mathrm{Ext}^1(P,-) = 0$ as the **(merely) projective** abelian groups, i.e. abelian groups P for which epimorphisms $A \to P$ **merely split**:

$$\|\sum_{s:\mathrm{Ab}(P,A)} p \circ s = \mathrm{id}_P\|$$

Using the long exact sequence above, we show that Ext^n can be computed via (merely) projective resolutions, when they exist.

[1] Note that these are in particular sets, not ∞-groups.

When interpreted into a Grothendieck ∞-topos, our constructions recover sheaf Ext [3] over the free ring on one generator $\mathbb{Z}$ in the 1-topos of 0-truncated objects (at a specified universe level), along with its long exact sequence. Examples of nontrivial higher Ext sheaves are known to algebraic geometers, and give nontrivial examples of our Ext^n in a model. As a concrete example, we show that the Sierpinski ∞-topos has **internal homological dimension** 2. In particular we construct a concrete example of a nontrivial Ext^2 sheaf.

The (merely) projective abelian groups in homotopy type theory are interpreted as **internally projective** sheaves of abelian groups in a Grothendieck ∞-topos. Familiar examples of these are the locally free sheaves of finite rank, and it is well-known that sheaf Ext may be computed via (finite) locally free resolutions. However, our concrete example in the Sierpinski ∞-topos is computed via an internal projective resolution that is not locally free. In continuing work on this project, we hope to discover conditions for the existence of enough internal projectives and whether such resolutions enable interesting new computations.

A defect of our construction is that Ext^n is a large type. This may seem inevitable, since there are known examples of abelian categories where the classical Yoneda Ext groups are not sets, but proper classes. But those are *external* Ext groups, whereas ours are *internal*. In fact, we prove that Ext^1 is always equivalent to a small type. More precisely, by combining results from [1] and [5], we construct equivalences

$$\mathcal{E}xt^1(B, A) \ \simeq\ \big(\mathrm{K}(B, n) \to_* \mathrm{BAut}(\mathrm{K}(A, n))\big), \qquad \text{for } n \geq 2$$

where the right-hand side is small by a theorem of Rijke [6]. Consequently $\mathcal{E}xt^1$ and Ext^1 are both equivalent to small types. Whether Ext^n is small for $n \geq 2$ is not yet known, but a potential line of attack is to study the structure of $\mathcal{E}xt^1(B, A)$ reflected onto $\mathrm{K}(B, 2) \to_* \mathrm{BAut}(\mathrm{K}(A, 2))$ via the above equivalence, in combination with the recursive formula for $\mathrm{Ext}^n(B, A)$.

Largeness of $\mathrm{Ext}^1(B, A)$ in an ∞-topos means that $\mathrm{Ext}^1(B, A)$ lives in the 1-topos of *large* 0-truncated objects, one universe level above B and A. However, one can always define sheaf Ext via injective resolutions in sheaf toposes, and there are no phenomena involving sizes of objects in that context. Since our Ext groups recover sheaf Ext, we already know Ext^n to be equivalent to a small type in these models. Our result about smallness of Ext^1 is therefore mainly of interest in more general models of homotopy type theory — examples of which are not yet well understood.

References

[1] Ulrik Buchholtz, Floris van Doorn, and Egbert Rijke. "Higher Groups in Homotopy Type Theory". In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science* (2018).

[2] René Guitart and Luc Van den Bril. "Calcul des satellites et présentations des bimodules à l'aide des carrés exacts (2e partie)". In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 24.4 (1983), pp. 333–369. URL: http://archive.numdam.org/item/CTGDC_1983__24_4_333_0/.

[3] Robin Hartshorne. *Algebraic geometry*. Springer, 1977.

[4] Saunders Mac Lane. *Homology*. Springer, 1963.

[5] David Jaz Myers. "Higher Schreier Theory". Slides from the HoTTEST Conference 2020. URL: http://davidjaz.com/Talks/DJM_HoTT2020.pdf.

[6] Egbert Rijke. *The join construction*. 2017. arXiv: 1701.07538.

[7] Michael Shulman. *All $(\infty, 1)$-toposes have strict univalent universes*. 2019. arXiv: 1904.07004.

[8] Andrew W Swan. *On the Nielsen-Schreier Theorem in Homotopy Type Theory*. 2020. arXiv: 2010.01187.

[9] *The Coq HoTT Library*. URL: https://github.com/HoTT/HoTT.

[10] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.