NON-ACCESSIBLE LOCALIZATIONS

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Motivation. In topology, the study of reflective subcategories (usually called "localizations") has a long history, beginning with work of Sullivan and Bousfield. This framework has played a fundamental organizing role in algebraic topology in the intervening decades, influencing and leading to the solution of many central conjectures in the field. More recently, the theory of reflective subcategories was adapted to the setting of ∞-categories by Lurie [Lur]. The key ingredient of a localization L is a function sending an object X to an object LX, with certain properties.

Independently, logicians and philosophers have considered the notion of modalities in logic, which allow one to qualify statements to express *possibility*, *necessity*, and other attributes such as *temporal* modalities. These modalities are expressed by applying a modal operator $\lozenge$ to a proposition P to produce a new proposition $\lozenge P$, with certain properties.

The notion of a reflective subuniverse in homotopy type theory simultaneously encodes the idea of a localization of an ∞-topos and common modalities studied in logic [Cor]. When interpreted in an ∞-category, a reflective subuniverse of a universe $\mathcal{U}$ in homotopy type theory corresponds to a family of reflective subcategories of each slice category, compatible with pullback (see [RSS, Appendix A] and [Ver]). In addition, reflective subuniverses appear in cohesive type theory, which can capture continuous and smooth geometry.

The foundations for the theory of reflective subuniverses in homotopy type theory were developed in [RSS]. We briefly describe the background from there needed to explain our results.

Background. All of our results are stated and proved within the framework of homotopy type theory. We follow the conventions and notation of [Uni]. All of the results have been formalized using the Coq HoTT library [HoTT].

Let $\mathcal{U}$ and $\mathcal{U}'$ be univalent universes with $\mathcal{U} : \mathcal{U}'$ and $\prod_{X:\mathcal{U}} X : \mathcal{U}'$. Unless otherwise specified, we assume that all types are in $\mathcal{U}'$.

Recall from [RSS, Uni] that a **reflective subuniverse** L of $\mathcal{U}$ consists of a subuniverse $\mathsf{is\text{-}local}_L : \mathcal{U} \to \mathsf{Prop}$, a function $L : \mathcal{U} \to \mathcal{U}$, and a **localization** $\eta_X : X \to LX$ for each $X : \mathcal{U}$. Being a localization means that LX is L-local and η_X is initial among maps whose codomain is local.

Consider a family f of maps in $\mathcal{U}$. More precisely, let $I : \mathcal{U}$, $A : I \to \mathcal{U}$, $B : I \to \mathcal{U}$ and $f : \prod_{i:I}(A_i \to B_i)$. A type X is ***f*-local** if precomposition with f_i

$$X^{B_i} \to X^{A_i}$$

is an equivalence for each $i : I$. A key result in the subject is [RSS, Theorem 2.18], which states that the subuniverse of f-local types in $\mathcal{U}$ is reflective. Such a reflective subuniverse is called **accessible**.

Given a reflective subuniverse L of $\mathcal{U}$, we can define a new subuniverse L' of $\mathcal{U}$ consisting of the types whose identity types are L-local. In [CORS], it is shown that L' is again reflective. In the proof, care was taken to avoid the assumption that L is accessible.

This raises the question of whether there are interesting non-accessible reflective subuniverses. In topology, work of [CSS] shows that there exists a reflective subcategory of the category of spaces whose accessibility is independent of ZFC. The only example I am aware of in type theory is the double-negation modality, which is a reflection onto the subuniverse of **stable propositions** (those P such that $P \to \neg\neg P$ is an equivalence). However, this is accessible if one assumes propositional resizing, which holds in models associated to ∞-toposes.

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Main result. The goal of the present work is to give a construction of a reflective subuniverse in homotopy type theory which when interpreted in spaces reproduces the example of [CSS] whose accessibility is independent of ZFC. Our result is in fact more general. First, it gives a result in any model of homotopy type theory (e.g., in any ∞-topos). Second, it also produces examples at higher truncation levels. Moreover, by working within HoTT, we are able to give a simpler proof, not at all following the classical proof. We do not (yet) have internal arguments showing when our examples cannot be proven to be accessible.

The main result is:

Theorem. *Let $n \geq -1$. Let $f : \prod_{i:I}(A_i \to B_i)$ be a family of $(n-1)$-connected maps in $\mathcal{U}'$. Then the subuniverse of n-truncated f-local types in $\mathcal{U}$ is reflective.*

The key point is that none of I, A_i or B_i are required to be in $\mathcal{U}$. They can be in any other universe. When $n = 1$, this theorem reproduces the classical result of [CSS]. When $n > 1$, this result appears to be new, even for the ∞-topos of spaces.

We remind the reader that a map $g : A \to B$ is said to be ***k*-connected** if for every $b : B$, the k-truncation of the fibre $\sum_{a:A} (f(a) = b)$ is contractible. For example, the (-1)-truncated maps are precisely the surjections.

Methods. In order to prove the main theorem, we first prove results that allow one to prove that a type is equivalent to a type in $\mathcal{U}$.

Say that a type X is **small** if it is equivalent to a type in $\mathcal{U}$. More precisely, if $\sum_{X':\mathcal{U}} X' \simeq X$. Note that this is a mere proposition, by univalence. Define X to be **0-locally small** if it is small. Define X to be **$(n+1)$-locally small** if for all $x, x' : X$, $x = x'$ is n-locally small. These are also mere propositions.

Using his join construction [R], Rijke showed that if A is small, X is 1-locally small and $A \to X$ is (-1)-connected (surjective), then X is small. Rijke's result is the $n = 0$ case of the following result:

Proposition. *Let $n \geq -1$. If $f : A \to X$ is $(n-1)$-connected, A is small and X is $(n+1)$-locally small, then X is small.*

To prove the main result, we consider the extended family $\bar{f}$, indexed by $I + 1$, which also includes the map $S^{n+1} \to 1$. The $\bar{f}$-local types are exactly the n-truncated types which are f-local. We show that localization with respect to $\bar{f}$ on $\mathcal{U}'$ restricts to a localization on $\mathcal{U}$, using the proposition and [CORS, Theorem 3.12]. In addition, in order to adjust the predicate $\mathsf{is\text{-}local}_{\bar{f}}$ to land in $\mathcal{U}$, we need to use **propositional resizing**, the statement that every mere proposition in $\mathcal{U}'$ is small.

Along the way, we prove other results about smallness, and also show that the extended family $\bar{f}$ generates an orthogonal factorization system on $\mathcal{U}$.

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