COPRODUCTS IN ∞-LCCCS WITH SUBOBJECT CLASSIFIER

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Abstract. Using a type theoretic argument, we show that every locally cartesian closed ∞-category with a subobject classifier has disjoint finite coproducts.

Early axiomatizations of elementary toposes postulated finite colimits [Law70, Tie72], but it was soon realized [Mik72, Par74] that their existence follows from the other axioms, namely finite limits, cartesian closure, and the existence of a subobject classifier.

The present work is motivated by the question if an analogous reduction is possible for the recently proposed notion of *elementary ∞-toposes* [Shu17, Ras18], and we give a partial positive answer by showing that locally cartesian closed ∞-categories (∞-LCCCs) with a subobject classifier have disjoint finite coproducts. The proof is outlined in the following.

For readability, the use of ∞-categories in this abstract is informal and 'model-independent'. We also employ type theory informally as an 'internal language' of ∞-LCCCs, but our use of type theory should be viewed only as a heuristic, as the arguments are small enough to be translated into category theoretic arguments 'by hand'.

The tripos of subobjects. A *subobject* of an object A in an ∞-category $\mathcal{C}$ is an embedding ((-1)-truncated map) $f : U \hookrightarrow A$. The subobjects of A form a preorder (i.e. an ∞-category in which all homs are propositions) $\mathsf{Sub}(A)$ and if $\mathcal{C}$ has pullbacks then the assignment $A \mapsto \mathsf{Sub}(A)$ is contravariant in A, giving rise to a functor

$$\mathsf{Sub} : \mathcal{C}^{\mathsf{op}} \to \mathbf{Ord}.$$

A *subobject classifier* (SOC) is an object Ω representing the presheaf of underlying types

$$\mathcal{C}^{\mathsf{op}} \xrightarrow{\mathsf{Sub}} \mathbf{Ord} \xrightarrow{\mathsf{Core}} \mathcal{S}.$$

Since the core of any preorder is 0-truncated, we conclude that Ω is 0-truncated. We denote the universal element of the representation by $(\mathsf{tt} : V \hookrightarrow \Omega) \in \mathsf{Sub}(\Omega)$; as in the case of 1-toposes one can show that its domain V is terminal. The subobject functor factors through the homotopy category of $\mathcal{C}$

$$\begin{array}{ccc} \mathcal{C}^{\mathsf{op}} & \xrightarrow{\mathsf{Sub}} & \mathbf{Ord} \\ \downarrow & \nearrow_{\mathsf{Sub}_0} & \\ \mathrm{Ho}(\mathcal{C})^{\mathsf{op}} & & \end{array}$$

and if $\mathcal{C}$ is locally cartesian closed and has a SOC, then it is straightforward to show that:

Lemma 1. Sub_0 *is a* tripos [HJP80]. *In particular the preorders* $\mathsf{Sub}(A)$ *then have finite joins which are constructed from Heyting implication and the SOC using the standard encodings in 2nd order logic*[1].

The initial object. The existence of initial objects in ∞-LCCCs with SOC is a consequence of the following lemma.

Lemma 2. *The following are equivalent for an object I of an ∞-LCCC $\mathcal{C}$.*

(1) I is initial. *(2)* $\mathcal{C}/I \simeq 1$ *(3)* $\mathsf{Sub}(I) \simeq 1$

Proof. It is easy to see that both 1 and 2 imply 3. Conversely, 1 and 2 follow from 3 by the derivations

$$\dfrac{\dfrac{i : I \vdash \mathrm{isContr}(A)}{\vdash \mathrm{isContr}(\Pi\, i : I \,.\, A)}}{A^I \text{ is contractible}} \qquad \dfrac{i : I \vdash \mathrm{isContr}(\Sigma a \,.\, f\, a = i)}{f : A \to I \text{ is an equivalence}}$$

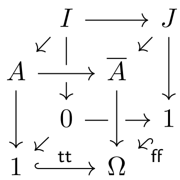
because the contractibility statements are propositions in context $i : I$. □

Corollary 3. *Any ∞-LCCC with SOC has an initial object.*

[1]Note that a subobject classifier in an LCCC is always an 'impredicative universe', since subobjects are closed under arbitary Π's.

Proof. Let $0 \hookrightarrow 1$ be the least subobject of 1 (which exists by Lemma 1). Then 0 can't have any non-trivial subobjects, and therefore is initial by Lemma 2. $\square$

Binary coproducts. Assume that $\mathcal{C}$ is an ∞-LCCC with SOC $\mathsf{tt} : 1 \hookrightarrow \Omega$. For every embedding $m : U \hookrightarrow V$ in $\mathcal{C}$, the adjunction $m^* \dashv \Pi_m : \mathcal{C}/U \to \mathcal{C}/V$ is a reflection. In particular, given $A \in \mathcal{C}$, we get a pullback square $\begin{array}{ccc} A & \longrightarrow & \overline{A} \\ \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow{\mathsf{tt}} & \Omega \end{array}$ by setting $(\overline{A} \to \Omega) = \Pi_{\mathsf{tt}}(A \to 1)^2$. Now let $\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{tt}} \\ 1 & \xrightarrow{\mathsf{ff}} & \Omega \end{array}$ be the classifying square of the least subobject $0 \hookrightarrow 1$ of 1. Then the classifying map ff is an embedding since all points of 0-types are. By the Beck-Chevalley condition, chasing $(A \to 1)$ around this square using pullback and pushforward yields a commutative cube



where all sides are pullbacks, $I \to 0$ is an equivalence by Lemma 2, and $J \to 1$ is an equivalence since Π-functors preserve terminal objects. Renaming I and J to 0 and 1 we get a pullback $\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ A & \hookrightarrow & \overline{A} \end{array}$ on top of the cube: we have embedded A into a larger object with a disjoint point. Given a second object B, we form the transposed product of the respective squares

$$\left(\begin{array}{ccc} 0 & \hookrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ A & \hookrightarrow & \overline{A} \end{array}\right) \times \left(\begin{array}{ccc} 0 & \hookrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ 1 & \hookrightarrow & \overline{B} \end{array}\right) = \left(\begin{array}{ccc} 0 & \hookrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ A & \hookrightarrow & \overline{A} \times \overline{B} \end{array}\right)$$

to obtain an object that embeds A and B disjointly. Forming the join of A and B in $\overline{A} \times \overline{B}$ yields a cospan $A \overset{i}{\hookrightarrow} C \overset{j}{\hookleftarrow} B$ such that $A \wedge B = \bot$ and $A \vee B = \top$ in $\mathsf{Sub}(C)$. It remains to show the following:

Lemma 4. *Let $A \overset{i}{\hookrightarrow} C \overset{j}{\hookleftarrow} B$ be a cospan of embeddings in an ∞-LCCC, such that $A \wedge B = \bot$ and $A \vee B = \top$ in $\mathsf{Sub}(C)$. Then i and j exhibit C as a coproduct of A and B.*

Proof. Given another cospan $A \overset{f}{\hookrightarrow} X \overset{g}{\hookleftarrow} B$ it suffices to show that the object described by the formula

$$\Sigma(h : C \to X) \,.\, (h \circ i = f) \times (h \circ j = g)$$

is contractible. This is provable type theoretically after rewriting the formula as

$$\Pi\, c\, \Sigma\, x \,.\, (\Pi\, a \,.\, c = i\, a \to x = f\, a) \times (\Pi\, b \,.\, c = j\, b \to x = g\, b)$$

using the type theoretic axiom of choice. $\square$

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[2] In classical topos theory, the object $\overline{A}$ is known as *partial map representer*, since maps $X \to \overline{A}$ correspond to partial maps from X to A [Joh02, A2.4]. However, whereas in 1-toposes $\overline{A}$ is a subobject of Ω^A (namely the object of 'sub-singletons'), in the higher setting the latter is always a 0-type whereas $\overline{A}$ embeds A.