Some simple higher rings

Felix Cherubini

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This work profited from discussions with Mathieu Anel in 2018 and 2019 and some more recent discussion with David Jaz Myers and Tobias Columbus.

In the following, a ring is a 0-type R, with the structure (i.e. $+$ and 0) and laws of an abelian group, together with a multiplication $\cdot : R \to R \to R$ and a unit $1 : R$ such that $(R, \cdot, 1)$ is a monoid and the usual distributivity laws hold. Rings are a central object of study in Algebra and also have a higher analogue, which we will call ∞-rings. It is not known how (or if) ∞-rings can be defined in plain HoTT. The point of this work is to look at some rather simple examples of ∞-rings, that can be constructed in HoTT.

One important way to construct new rings from given ones, is to take quotients. Usually, quotients are defined with respect to a twosided ideal, which is a subring closed under multiplication with arbitrary ring elements from both sides. This is however not a good starting point for generalization to ∞-rings. Instead of looking at a subset, we need to look at morphisms into a ring. A principal ideal, i.e. the subring generated by a single element $\alpha : R$ can be written as the image of the morphism of the underlying abelian groups

$$\alpha \cdot \_ : R \to R$$

The underlying abelian group of the quotient ring $R/(\alpha)$ is then the cokernel of this morphism. This translates directly to higher rings, where the cokernel of abelian ∞-groups is given as a homotopy quotient.

Fortunately, as the author learned from David Jaz Myers, this homotopy quotient is surprisingly easy to construct in the following, commonly known way to define abelian ∞-groups. An ∞-group can be defined as a 1-connected pointed type $(BG, *)$, since that determines the type $G :\equiv \Omega BG = (* =_{BG} *)$, which carries the coherent structure of a higher group. We can ask for higher deloopings $B^n G$ of G, where $B^n G$ has to be a pointed n-connected type with a pointed equivalence $(* =_{B^n G} *) \simeq B^{n-1} G$. An ∞-group with infinitely many deloopings is called abelian.

A morphism $f : A \to A'$ of abelian ∞-groups A, A' is a sequence of pointed maps between the deloopings, that commute with the structure equivalences:

$$\begin{array}{ccc} B^n A & \xrightarrow{f_n} & B^n A' \\ \downarrow & & \downarrow \\ \Omega B^{n+1} A & \xrightarrow[\Omega f_{n+1}]{} & \Omega B^{n+1} A' \end{array}$$

Kernels of these morphisms can be computed pointwise:

$$\mathrm{Ker}(f)_n :\equiv \mathrm{fib}_{f_n}$$

And Cokernels in the same way by shifting:

$$\mathrm{Coker}(f)_n :\equiv \mathrm{fib}_{f_{n+1}}$$

The map from A' to $\mathrm{Coker}(f)$ is given levelwise by pasting pullback squares:

$$\begin{array}{ccccccc} & & 1 & \longrightarrow & B^{n+1}A & \xrightarrow{f_{n+1}} & B^{n+1}A' \\ & & \uparrow & & \uparrow & & \uparrow \\ B^nA' & \xrightarrow{\simeq} & \Omega B^{n+1}A' & \longrightarrow & \mathrm{fib}_{f_{n+1}} & \longrightarrow & 1 \end{array}$$

Going back to rings, we can ask for a multiplication on an abelian ∞-group. So for an abelian ∞-group, let νA be ΩBA and for any $\alpha : \nu A$, let

$$\alpha \cdot \_ : A \to A \text{ and } \_ \cdot \alpha : A \to A$$

be morphisms of abelian ∞-groups. We can ask for an 1-associative multiplication by asking for a homotopy for all $\alpha, \beta : \nu A$ in the square:

$$\begin{array}{ccc} A & \xrightarrow{\alpha\cdot\_} & A \\ {\scriptstyle \_\cdot\beta}\downarrow & & \downarrow{\scriptstyle \_\cdot\beta} \\ A & \xrightarrow[\alpha\cdot\_]{} & A \end{array}$$

And, of course, we can encode other laws we want for the rings to hold as commutativity of diagrams. It turns out, that all the structure and diagrams of a 1-ring decend to the cokernel of a map $f : A \to A'$, if it commutes with left- and right-multiplication, i.e. if the following squares commute: This is not so much a surprise, since we constructed the cokernel levelwise as a pullback and all the laws and structures are induced by the universal property of the pullbacks. The author therefore expects, that given higher coherences decend to cokernels as well.
So far, this could be summarized as: We have a type of rings which have a coherent group structure, but not necessarily a coherent multiplication. Let us call these rings semi-wild. We can take quotients of these rings and expect, that this construction preserves any further coherences a semi-wild ring might have. Any 1-ring A can be mapped to a semi wild ring ιA, by taking the sequence of pointed types $K(A, n)$ together with the morphisms and diagrams induced by the 1-ring structure. These rings trivially satisfy higher coherence laws. It is also possible to calculate the identity types of quotients of ιA and they match what should be the case for simplicial rings.
The setup is easy enough, that examples of higher rings can be constructed as quotients and are understandable. We will show simple non-trivial examples where we can see how algebraic information is stored in the identity types of a quotient. We will also explain, how the more complicated operation of taking tensor products of rings is possible in this setup.