$(\infty, 1)$-Categorical comprehension schemes

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Comprehension schemes arose as crucial notions in the early work on the foundations of set theory, and hence found expression in a large variety of foundational settings for mathematics. Particularly, they have been introduced to the context of categorical logic first by Lawvere and then by Bénabou in the 1970s. Hence, they have been studied in different forms and have been applied to many examples throughout the literature of category theory. The expressive power of comprehension schemes encompasses important categorical constructions, most notably necessary and sufficient conditions to characterize locally cartesian closed categories and elementary toposes.

In his critique of the foundations of naive category theory [1], Bénabou provided an intuition to define the notion of comprehension schemes not only in category theory (over the topos of sets), but in category theory over any other category in a syntax-free way. In this generality, comprehension schemes become properties of Grothendieck fibrations over arbitrary categories. Which particular comprehension schemes are satisfied by a given Grothendieck fibration $\mathcal{E} \to \mathcal{C}$ then depends on the categorical constructions available in $\mathcal{E}$ over $\mathcal{C}$ and in $\mathcal{C}$ itself. The notion has been made precise in considerable generality by Johnstone in [2, B1.3], tieing together the elementary examples given in the glossary of [1] to a structurally well behaved theory.

In this talk – based on the synonymous paper [3] – we generalize Johnstone's notion of comprehension schemes for ordinary fibered categories and define a respective notion for cartesian fibrations over $(\infty, 1)$-categories. This includes natural generalizations of smallness, local smallness and the notion of definability in the sense of Bénabou.

It turns out that many characterizations of ordinary categorical constructions carry over flawlessly, while others in fact work out much better for the reason that "evil" equalities naturally arising in the context of ordinary category theory are implicitly replaced by "good" instances of equivalences between $(\infty, 1)$-categories. This also affects the remarks on the "strangeness" of equality in [1, §8], because commutativity of squares in $(\infty, 1)$-categories is a matter of equivalence, not of meta-mathematical equality. In this sense, the study of equality becomes a study of equivalence.

The aim of the talk will be to present some of the central results in [3] and apply them to interesting examples arising in higher topos theory, and more generally, the semantics of Homotopy Type Theory.

For instance, given an $(\infty, 1)$-category $\mathcal{C}$ with pullbacks, we can define the $(\infty, 1)$-categorical version of the externalization functor which takes an internal $(\infty, 1)$-category $\mathcal{X}$ in $\mathcal{C}$, and gives the "externalized" $\mathcal{C}$-indexed $(\infty, 1)$-category constructed from $\mathcal{X}$. In the paper, we show that the essential image of the externalization functor is exactly given by the small and locally small $\mathcal{C}$-indexed $(\infty, 1)$-categories. One may therefore think of such indexed $(\infty, 1)$-categories as the representable ones. This is a result which fails in the 1-categorical situation. A central example will be the universal cartesian fibration (defined in [4, 3.3.2]) which turns out to be represented (in this directed sense) by the "freely walking chain" $\Delta^{\bullet}$ in the $(\infty, 1)$-category of small $(\infty, 1)$-categories.

In the context of comprehension $(\infty, 1)$-categories à la Jacobs, one sees that the small comprehension $(\infty, 1)$-categories over a locally cartesian closed $(\infty, 1)$-category $\mathcal{C}$ stand in 1-1 correspondence to univalent maps in $\mathcal{C}$. It follows that one may characterize the elementary ∞-toposes in the sense of Shulman (and subsequently Rasekh) as those $(\infty, 1)$-categories which are locally small over themselves and exhibit a cover of small neighbourhoods.

In the context of the model categorical semantics of HoTT, the characterization of univalent maps as the small comprehension categories over a locally cartesian ∞-category $\mathcal{C}$ recovers notions well-known to the HoTT-literature.

Namely, it turns out that the small and locally small "indexed quasi-categories" over a given logical model category $\mathcal{M}$ are exactly the right Quillen functors from $\mathcal{M}^{op}$ to the Joyal model structure on simplicial sets. In the paper we show that various Quillen functors in the literature on homotopical algebra arise in this way, that is, by externalization of a complete Segal object. We further show that the externalization of the function type $\mathrm{Fun}(p)$ associated to a fibration $p \in \mathcal{M}$ with fibrant base is right Quillen if and only if p is univalent in $\mathcal{M}$. In fact, given a fibration $p \in \mathcal{M}$, whether or not it induces a right Quillen functor to the Joyal model structure can be reduced to a 2-dimensional (rather than an ∞-dimensional) lifting problem. As a consequence, univalence of p and fibrancy of B can be checked by a right Quillen condition into the Lack model structure on the category of 2-categories. This simpler right Quillen condition corresponds precisely to the conjunction of what has become known as the weak equivalence extension property and the fibration extension property of p (or, rather, of the class of arrows which arise as pullback of p). We therefore obtain an "external" explanation of the meaning of these two conceptually rather opaque properties.

References

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