Globular weak ω-categories as models of a type theory

Thibaut Benjamin*, Eric Finster† and Samuel Mimram‡

Introduction. Weak ω-categories are a notoriously difficult notion to define and work with, as they require very recursive axioms. In order to solve this problem, two of the authors have proposed a new approach to these structures [5] by taking advantage of recent development in computer-aided mathematics, in the form of a type theory called CaTT. The precise relation with previously understood notions of weak ω-categories was however yet to be established, and left as a conjecture [5, Conjecture 49]. Our main goal here is to briefly present a proof of this conjecture that appears in [3]. This also proves a particular case of the initiality conjecture.

Homotopy type theory and weak ω-groupoids. One of the main takeaway of the developments about Martin-Löf type theory from the 2000's is that the identity types are note as simple as they were originally thought to be. The first manifestation of this phenomenon is given by the 2-groupoid model, due to Hofmann and Streicher [7], showing that identity types between identity types actually contain relevant computational information. This fact that then strengthened to add higher structure and culminated in a correspondence between the types and weak ω-groupoids [8, 11, 1], a structure with data in infinitely many levels. The prevalent role of weak ω-groupoids in homotopy theory, most notably via the homotopy hypothesis [6], linked type theory to the study of spaces, which is now known as homotopy type theory [10].

Weak ω-groupoids as an algebraic structure. The study of weak ω-groupoids is anterior to its connection to type theory, and is very rich, and covering any substantial part is beyond the scope of this abstract. One of the definitions of interest for our purpose is due to Grothendieck [6], which is based on a globular shape. A precise way to understand the connection with homotopy theory was introduced by Brunerie [4], by stripping homotopy type theory off of all the rules but the ones that generate the weak ω-category structure, and proving that the resulting type theory is equivalent to the Grothendieck definition of weak ω-groupoids.

Grothendieck-Maltsiniotis weak ω-categeories Weak ω-categories are higher structures akin to weak ω-groupoids, but in which every cell is endowed with a preferred direction. By leveraging the similarities between these structures, Maltsiniotis proposed a definition of those, based on the Grothendieck definition of weak ω-groupoids. Since this definition is quite technical, we simply give here a rough sketch of its main steps:

- Identify the pasting schemes [2, 9]: They are the globular sets destined to be an essentially unique operation, and assemble into a category Θ_0.

*CEA LIST
†University of Cambridge
‡Laboratoire d'informatique de l'Ecole Polytechnique

- Construct the category Θ_∞ by freely adding morphisms witnessing the required structure to Θ_0. This exension is characterized by a universal property.

- Construct weak ω-categories by gluing elements of Θ_∞. Formally this defines the weak ω-categories as the presheaves $\Theta_\infty^{\mathrm{op}} \to \mathbf{Set}$ that preserve the pasting schemes.

The type theory CaTT The proposal of CaTT can be understood as the pushout of these two variations of Grothendieck's approach to weak ω-categories.

$$\begin{tikzcd}
\begin{array}{c}\text{Grothendieck weak}\\ \omega\text{-groupoids [6]}\end{array} \arrow[r, "\text{directionality}"] \arrow[d, "\text{type theory}"] & \begin{array}{c}\text{Grothendieck-Maltsiniotis weak}\\ \omega\text{-cateogries [9]}\end{array} \arrow[d, "\text{type theory}"] \\
\text{Brunerie's type theory [4]} \arrow[r, "\text{directionality}"] & \mathsf{CaTT}\ \text{[5]}
\end{tikzcd}$$

In this theory, the type constructors generate the combinatorics of globular sets. There is a judgment $\Gamma \vdash_{\mathsf{ps}}$ to recognize the pasting schemes, and the term constructors use this judgment to represent the operations and axioms of weak ω-categories.

Models of a type theory The contexts of the theory CaTT assemble into a category $\mathcal{S}_{\mathsf{CaTT}}$, whose morphisms are the substitutions. This category is equipped with additional structure that mirror the structure of the type theory called a *category with families* (CwF). The category **Set** of sets is also a CwF (up to size issues) and a *model* of the type theory CaTT is a morphism of CwF $\mathcal{S}_{\mathsf{CaTT}} \to \mathbf{Set}$.

Models of the theory CaTT

Theorem. *There is an equivalence of categories between the models of the theory CaTT and the Grothendieck-Maltsiniotis weak ω-categories.*

Sketch of proof. The proof is done in three steps:

- Define the full subcategory $\mathcal{S}_{\mathsf{PS}}$ of $\mathcal{S}_{\mathsf{CaTT}}$ by using the judgment $\Gamma \vdash_{\mathsf{ps}}$. An argument of universal property shows
$$\mathcal{S}_{\mathsf{PS}} \simeq \Theta_\infty^{\mathrm{op}} \tag{1}$$

- Model of CaTT restrict along the subcategory inclusion into $\mathcal{S}_{\mathsf{PS}} \to \mathbf{Set}$. This functor defines a weak ω-category by (1) and thanks to the CwF structure.

- Conversely the right Kan extension along the subcategory inclusion $\Theta_\infty^{\mathrm{op}} \to \mathbf{Set}$ given by (1) provides a model of CaTT for each weak ω-category.

□

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