Modal Fracture of Higher Groups

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May 27, 2021

To give a home for calculations done in [3], Cheeger and Simons [2] gave a series of lectures in 1973 defining and studying *differential characters*, which equip classes in ordinary integral cohomology with explicit differential form representatives. Slightly earlier, Deligne [4] had put forward a cohomology theory in the complex analytic setting which would go on to be called Deligne cohomology. It was later realized that when put in the differential geometric setting, Deligne cohomology gave a presentation of the theory of differential characters. This combined theory has become known as *ordinary differential cohomology*.

The ordinary differential cohomology $D_k(X)$ of a manifold X is characterized by its relationship to the ordinary cohomology of X and the differential forms on X by a diagram known as the *differential cohomology hexagon* or the *character diagram* [8]:

$$
\begin{array}{ccccccc}
 & & \Lambda^k(X)/\mathrm{im}(d) & \xrightarrow{\quad d \quad} & \Lambda^{k+1}_{\mathrm{cl}}(X) & & \\
 & \nearrow & & \searrow \quad \nearrow & & \searrow & \\
H^k(X;\mathbb{R}) & & & D_k(X) & & & H^{k+1}(X;\mathbb{R}) \\
 & \searrow & & \nearrow \quad \searrow & & \nearrow & \\
 & & H^k(X;U(1)) & \xrightarrow[\quad \beta \quad]{} & H^{k+1}(X;\mathbb{Z}) & &
\end{array}
\tag{1}
$$

In this diagram, the top and bottom sequences are long exact, and the diagonal sequences are exact in the middle. The bottom sequence is the Bockstein sequence associated to the universal cover short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$$

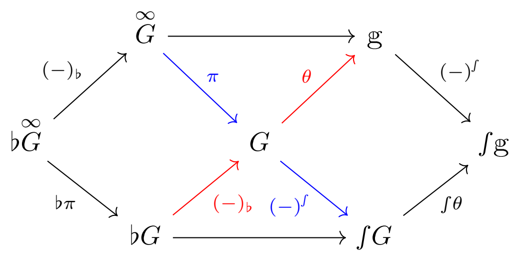
while the top sequence is given by de Rham's theorem representing real cohomology classes by differential forms.

This sort of diagram is characteristic of differential cohomology theories in general. Bunke, Nikolaus, and Vokel [1] interpret differential cohomology theories as sheaves of spectra on the site of smooth manifolds and show that any such sheaf gives rise to hexagons like Eq. (1).

In his book *Differential cohomology in a cohesive ∞-topos* [5], Schreiber shows that the hexagons of Bunke, Nikolaus, and Vokel arise from an adjoint pair $\int \dashv \flat$ of a modality $\int$ and a comodality $\flat$. In the case of differential cohomology, these are the operations sending a stack X on manifolds to it's $\mathbb{R}$-localization $\int X$ (it's "homotopy type") and to its homotopy type of global sections $\flat X$, both considered as locally constant stacks. Having rendered these hexagons in purely modal terms, Schreiber argues in [6] that they should be thought of as *fracture squares*, and that indeed the traditional fracture squares are induced by adjoint (co)modalities in this way.

In this talk, we will construct the *modal fracture hexagon* of a higher group in Shulman's cohesive homotopy type theory [7] (or, rather, it's "strongly ∞-connected" component consisting of $\int$ and $\flat$ only). That is, we prove the following theorem:

Theorem 1. *Let G be a crisp ∞-group. Then we have the* modal fracture hexagon*:*



where

- $\theta : G \to \mathrm{g}$ *is the* infinitesimal remainder *of G, the quotient $G /\!\!/ \flat G$, and*
- $\pi : \overset{\infty}{G} \to G$ *is the* universal (contractible) ∞-cover *of G.*

Moreover,

1. *The middle diagonal sequences are fiber sequences.*
2. *The top and bottom sequences are fiber sequences.*
3. *Both squares are pullbacks.*

Furthermore, the homotopy type of g is a delooping of $\flat\overset{\infty}{G}$:

$$\int \mathrm{g} = \flat \boldsymbol{B}\overset{\infty}{G}.$$

Therefore, if G is k-commutative for $k \geq 1$ (that is, admits futher deloopings $\boldsymbol{B}^{k+1}G$), then we may continue the modal fracture hexagon on to $\boldsymbol{B}^k G$.

References

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