Univalent transport and type-theoretic ∞-categories

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Univalent transport Voevodsky's univalence axiom provides a way to transport properties and structures that are expressed in the language of type theory over isomorphisms and equivalences. It can be informally justified by the fact that all type-theoretic constructions preserve equivalences. However this fact also holds for non-univalent type theories, and even for some type theories that are inconsistent with univalence. *Univalent transport* is an external counterpart to the internal univalence axiom that can be proven for these theories.

In [5, 6], univalent transport is proven for type theories with identity types, Σ-types and Π-types. It is a consequence of the construction of left semi-model structures on the categories of models of these type theories. In [8], a *univalent parametricity* translation is defined, proving univalent transport for an univalent type theory. While univalent transport trivially holds for an univalent type theory, it is often the case that the univalence axiom does not occur in the terms obtained by this translation. It is then possible to use these terms in a non-univalent type theory. Both [5, 6] and [8] rely on some *homotopical inverse diagram* models of type theory.

In this work we introduce new formulations of univalent transport in the language of *type-theoretic ∞-categories*. We give new proofs of univalent transport that are valid even for type theories that may be inconsistent with univalence. We also prove completeness results that extend the applicability of these results.

Type-theoretic 1-categories and ∞-categories Some type theories are known or conjectured to be the internal languages of structured 1- and ∞- categories. For example, type theory with equality, Σ-, $\mathbf{1}$- and Π- types corresponds to locally cartesian closed 1-categories [2, 3]; and type theory with identity types, Σ- and $\mathbf{1}$- types corresponds to finitely complete ∞-categories [7].

This means that any model of type theory with Eq, $\mathbf{1}$ and Σ can be seen as a finitely complete 1-category, and so on. We use models of type theories instead of any other definition of these structured ∞-categories (such as quasicategories), and call them *type-theoretic ∞-categories*. For instance a type-theoretic finitely complete ∞-category $\mathcal{C} : \mathbf{CwF}^{\mathsf{lex}_\infty}$ is a model of type theory (CwF) equipped with Id, $\mathbf{1}$ and Σ.

We use the abbreviations lex_1, rep_1 and lcc_1 to refer respectively to finitely complete 1-categories, representable map 1-categories and locally cartesian closed 1-categories. We use lex_∞, rep_∞ and lcc_∞ to refer to the similarly structured ∞-categories.

We also have theories $\mathsf{lex}_{\mathsf{pre}}$, $\mathsf{rep}_{\mathsf{pre}}$ and $\mathsf{lcc}_{\mathsf{pre}}$ with neither equality or identity types. They do not directly correspond to categories; but they can be used to compare type-theoretic 1-categories and ∞-categories.

Internal models and generalized syntax The notion of model of a type theory $\mathbb{T}$ can be specified internally to presheaf categories. This only relies on the fact that presheaf categories are representable map 1-categories [9]; we have in fact a notion of internal model of $\mathbb{T}$ internally to any representable map 1-category.

Thus we can consider the initial object $\mathcal{S}yn_{\mathbb{T}}^{\mathsf{rep}_1}$ among type-theoretic representable 1-categories equipped with an internal model of $\mathbb{T}$. Similarly, we can consider $\mathcal{S}yn_{\mathbb{T}}^{\mathsf{lcc}_1}$, $\mathcal{S}yn_{\mathbb{T}}^{\mathsf{rep}_{\mathsf{pre}}}$, $\mathcal{S}yn_{\mathbb{T}}^{\mathsf{lcc}_{\mathsf{pre}}}$, $\mathcal{S}yn_{\mathbb{T}}^{\mathsf{rep}_\infty}$, $\mathcal{S}yn_{\mathbb{T}}^{\mathsf{lcc}_\infty}$, etc.

These *generalized syntactic* models are often useful; for instance the objects of $\mathcal{S}yn_{\mathbb{T}}^{\mathsf{rep}_1}$ correspond exactly to the finitely presented models of $\mathbb{T}$, whereas the objects of $\mathcal{S}yn_{\mathbb{T}}^{\mathsf{rep}_{\mathsf{pre}}}$ correspond to the finitely generated models.

Univalent models In a type-theoretic representable map (or locally cartesian closed) ∞-category $\mathcal{C} : \mathbf{CwF}^{\mathsf{rep}_\infty}$ equipped with an internal model $\mathcal{M}$ of a type theory $\mathbb{T}$, we have several internal notions of equality for the types and terms of $\mathcal{M}$. Two types (or terms) can be compared up to equivalence (or typal equality) in $\mathcal{M}$, or up to typal equality in $\mathcal{C}$. We say that the internal model $\mathcal{M}$ is univalent when these two notions coincide, i.e. when the canonical maps

$$\begin{aligned} \mathsf{coe}_{\mathsf{ty}} &: (A \simeq^o_{\mathsf{ty}} B) \to (A \cong B), \\ \mathsf{coe}_{\mathsf{tm}} &: (a \simeq^o_{\mathsf{tm}(A)} b) \to (A \simeq B) \end{aligned}$$

are equivalences in $\mathcal{C}$, where $(- \simeq^o -)$ is outer typal equality, $(- \cong -)$ is inner equivalence and $(- \simeq -)$ is inner typal equality.

We have an initial object $\mathcal{S}yn_{\mathbb{T},\mathsf{univ}}^{\mathsf{rep}_\infty}$ among type-theoretic representable map ∞-category equipped with an univalent internal model of $\mathbb{T}$. We can see $\mathcal{S}yn_{\mathbb{T},\mathsf{univ}}^{\mathsf{rep}_\infty}$ as the (type-theoretic) representable map ∞-category of the finitely generated models of $\mathbb{T}$. Similarly, we have a type-theoretic locally cartesian closed ∞-category $\mathcal{S}yn_{\mathbb{T},\mathsf{univ}}^{\mathsf{lcc}_\infty}$ equipped with an internal model of $\mathbb{T}$.

We say that $\mathbb{T}$ satisfies **univalent transport** when $\mathcal{S}yn_{\mathbb{T}}^{\mathsf{rep}_{\mathsf{pre}}} \to \mathcal{S}yn_{\mathbb{T},\mathsf{univ}}^{\mathsf{rep}_\infty}$ is essentially surjective on outer terms.

Completeness The following adequacy theorem is due to Gratzer and Sterling [4].

Theorem ([4])**.** *Let Σ be a presentation of representable map category $\mathcal{J}_\Sigma$ and let $\mathcal{E}_\Sigma$ be the locally cartesian closed category presented by the same data. Then the canonical functor $\mathcal{J}_\Sigma \to \mathcal{E}_\Sigma$ is fully faithful.*

Up to the equivalence between representable map categories and type-theoretic representable map categories, this equivalently says that for any type theory $\mathbb{T}$, the CwF morphism $\mathcal{S}yn_{\mathbb{T}}^{\mathsf{rep}_1} \to \mathcal{S}yn_{\mathbb{T}}^{\mathsf{lcc}_1}$ is bijective on outer terms.

This is also related to conservativity results for two-level type theory [1]. This can be generalized to other morphisms between type-theoretic 1-categories.

Theorem. *All of the morphisms in the following square are bijective on outer terms.*

$$\begin{array}{ccc} \mathcal{S}yn_{\mathbb{T}}^{\mathsf{rep}_{\mathsf{pre}}} & \longrightarrow & \mathcal{S}yn_{\mathbb{T}}^{\mathsf{lcc}_{\mathsf{pre}}} \\ \downarrow & & \downarrow \\ \mathcal{S}yn_{\mathbb{T}}^{\mathsf{rep}_1} & \longrightarrow & \mathcal{S}yn_{\mathbb{T}}^{\mathsf{lcc}_1} \end{array}$$

We generalize these completeness results to type-theoretic ∞-categories.

Theorem. *All of the morphisms in the following diagram are essentially surjective on outer terms.*

$$\begin{array}{ccc} \mathcal{S}yn_{\mathbb{T}}^{\mathsf{rep}_{\mathsf{pre}}} & & \mathcal{S}yn_{\mathbb{T}}^{\mathsf{lcc}_{\mathsf{pre}}} \\ \downarrow & & \downarrow \\ \mathcal{S}yn_{\mathbb{T},\mathsf{univ}}^{\mathsf{rep}_\infty} & \longrightarrow & \mathcal{S}yn_{\mathbb{T},\mathsf{univ}}^{\mathsf{lcc}_\infty} \end{array}$$

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