Coherence via ∞-type theories

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We prove Kapulkin and Lumsdaine's conjecture [3] that Martin-Löf type theory with a unit type, Σ-types, and intensional identity types gives *internal languages* for ∞-categories with finite limits. The techniques developed here will also be useful for formulating and proving other internal language conjectures.

One of the ultimate goals of homotopy type theory is to show that intensional type theory with univalence and higher inductive types gives internal languages for ∞-categories with certain structure. This should be formulated in some form of equivalence between theories and structured ∞-categories: given a structured ∞-category, one constructs a theory called the *internal language*; given a theory, one constructs an ∞-category called the *syntactic ∞-category*. The internal language of an ∞-category provides a syntactic way of reasoning about the ∞-category. Replacing interpretations of a theory by functors from the syntactic ∞-category, we have elegant ∞-categorical proofs of metatheorems about the theory.

Kapulkin and Lumsdaine's conjecture is the first step to this goal and asserts that theories over Martin-Löf's intensional type theory, without univalence or higher inductive types, are equivalent in a suitable sense to ∞-categories with finite limits. Although a version of their conjecture was proved by Kapulkin and Szumiło [4], the original one has still been open. The main difficulty in their conjecture is the *coherence problem*, a mismatch between levels of equality: in ∞-categories, most equations hold only up to *homotopy*, but type theories have a stricter notion of equality, *judgmental equality*.

The central idea of our approach is to introduce a notion of an *∞-type theory* which acts like a "bridge" between type theories and ∞-categories. Intuitively, an ∞-type theory is a kind of type theory in which judgmental equality is replaced by *homotopy* so that there is no coherence problem between ∞-type theories and ∞-categories. Ordinary type theories are special ∞-type theories in which homotopies are strict. Therefore, ∞-type theories are in between type theories and ∞-categories and have close connections to both of them.

We develop a general technique for internal language conjectures. Let us consider the example of Kapulkin and Lumsdaine's conjecture. Let $\mathbb{I}$ denote Martin-Löf's intensional type theory. We first introduce an ∞-type theory $\mathbb{E}_\infty$ which is an ∞-analogue of Martin-Löf's *extensional* type theory. One can show that theories over $\mathbb{E}_\infty$ are equivalent to ∞-categories with finite limits, which is an ∞-categorical analogue of Clairambault and Dybjer's result [2]. Then the problem is now how $\mathbb{I}$ and $\mathbb{E}_\infty$ are connected. This is as difficult as the original problem, but is formulated entirely in the language of ∞-type theories and related concepts. From a general theory of ∞-type theories, one can construct a functor

$$F : \mathbf{Th}(\mathbb{I}) \to \mathbf{Th}(\mathbb{E}_\infty),$$

where $\mathbf{Th}(T)$ is an ∞-category of theories over an ∞-type theory T. Kapulkin and Lumsdaine's conjecture is proved as follows:

1. observe that the functor $F : \mathbf{Th}(\mathbb{I}) \to \mathbf{Th}(\mathbb{E}_\infty) \simeq \mathbf{Lex}_\infty$ coincides with the one considered by Kapulkin and Lumsdaine [3, Conjecture 3.7], where $\mathbf{Lex}_\infty$ is the ∞-category of ∞-categories with finite limits;

2. show that the functor F exhibits $\mathbf{Th}(\mathbb{E}_\infty)$ as the localization of $\mathbf{Th}(\mathbb{I})$.

We note that the construction of the functor F also works when the (∞-)type theories are extended by type constructors such as Π-types, higher inductive types, and universes. Therefore, we can systematically formulate a family of internal language conjectures for extensions of intensional type theory.

We still have to solve some form of coherence problem to prove that the functor F is a localization. In our formulation, the coherence problem is solved by replacing a given model of $\mathbb{E}_\infty$ by a model of $\mathbb{I}$. We use Shulman's result on the existence of strict universes in (presentations of) ∞-toposes [5]. In view of natural models [1], a model of $\mathbb{I}$ is a strict universe in a presheaf topos. Analogously, a model of $\mathbb{E}_\infty$ is a (non-strict) universe in a presheaf ∞-topos. Given a model of $\mathbb{E}_\infty$, we replace the corresponding universe by some strict universe to obtain a model of $\mathbb{I}$ that approximates the given model of $\mathbb{E}_\infty$.

References

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