Effective Kan fibrations in Simplicial Sets

HoTT-UF 2020 extended abstract

Eric Faber*

(joint work with Benno van den Berg)

June 16, 2020

Introduction

The aim of this work is to develop an effective theory of Kan fibrations in simplicial sets, from scratch. Our motivation comes from earlier work of Bezem, Coquand and Parmann [BCP15] which shows that ordinary simplicial homotopy theory is ill-suited for providing a constructive model of dependent (homotopy) type theory. In our new approach, we define effective Kan fibrations for categories endowed with a *Moore structure*. Our current results can be summarised as follows:

1. Effective Kan fibrations for categories with Moore structure are closed under push forward;
2. The category of simplicial sets can be equipped with Moore structure;
3. In simplicial sets, effective Kan fibrations are a local notion of fibred structure;
4. Effective Kan fibrations admit fillers with respect to horn inclusions, hence they are ordinary Kan fibrations;
5. In a classical metatheory, any ordinary Kan fibration can be given the structure of an effective Kan fibration.

In the talk, I will give an overview of our paper, which is divided into two parts. The first part shows how to construct two algebraic weak factorisation systems (in the sense of [BG16]) in categories with Moore structure: one based on a class of cofibrations forming a *dominance*, and one based on Moore structure. These two are combined to define the notion of effective Kan fibration. The second part of the paper first proves the axioms of a Moore structure for the Moore path functor M that was first defined in [BG12]. Then, we show that the effective Kan fibrations present a local notion of fibred structure.

* eef25 'at' cantab.ac.uk

1 Effective Kan fibrations for categories with Moore structure

Definition 1. Let $\mathcal{E}$ be a category with finite limits. A *Moore structure* on $\mathcal{E}$ consists of the following:

(i) An endofunctor $M : \mathcal{E} \to \mathcal{E}$ which preserves pullbacks, together with natural transformations s,t,r,μ as in:

$$MX \times_{(t,s)} MX \xrightarrow{\mu_X} MX \underset{s_X}{\overset{t_X}{\rightrightarrows}} X, \qquad X \xrightarrow{r_X} MX$$

turning every X into an internal category, whose arrows are called *Moore paths*.

(ii) A natural transformation

$$\Gamma : MM \Rightarrow M, \qquad \text{and a } \textit{strength} \qquad \alpha_{X,Y} : X \times MY \to M(X \times Y)$$

respectively turning (M, s, Γ) into a comonad, and making all previous structure strong.

(iii) Several axioms, which establish Γ as a connection, or path contraction. Our axioms are stronger than the path object categories of [BG12]. Most notably, we demand a *distributive law*, saying:

$$\Gamma.\mu = \mu.(M\mu.\nu_X.(\Gamma.p_1, \theta_{MX}.\alpha_{MX,1}.(p_2, M!.p_1)), \Gamma.p_2),$$

where ν_X, θ_X are certain suitable isomorphisms.

Theorem 1. *A category with Moore structure defines an algebraic weak factorisation system (AWFS) in the sense of Bourke-Garner [BG16], factoring any $f : A \to B$ as follows:*

$$A \xrightarrow{(r.f,1)} MB \times_B A \xrightarrow{s.p_1} B.$$

The coalgebras for this AWFS are precisely coalgebras for the comonad Γ *and are called* hyperdeformation retracts *(HDRs).*

HDRs are bifibred over the domain functor dom : **HDR** $\to \mathcal{E}$.

Definition 2. Assume $\mathcal{E}$ comes equipped with a dominance, defining another AWFS whose left class consists of *cofibrations*. A *mould square* is a cartesian morphism of HDRs

$$\begin{array}{ccc} A' & \xrightarrow{i'} & B' \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle m'} \\ A & \xrightarrow{i} & B \end{array} \tag{1}$$

where m, m' are cofibrations.

For brevity, we now restrict to *symmetric* Moore structures (like for simplicial sets), but the results are more general.

Definition 3. An *effective Kan fibration* is a morphism $p : Y \to X$ equipped with a family of fillers $\phi_{a,b}(f, g, h)$ against mould squares as in:

$$\begin{array}{ccccc} A' & \xrightarrow{\quad i' \quad} & A & \xrightarrow{\quad a \quad} & Y \\ {\scriptstyle m}\downarrow & \forall h & \downarrow{\scriptstyle m'} & \exists \phi_{a,b}(f,g,h) & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[\quad i \quad]{} & B & \xrightarrow[\quad b \quad]{} & X \end{array},$$

satisfying appropriate compatibility conditions. These conditions can be formulated as having a right lifting structure with respect to a *triple category* of mould squares.

Theorem 2. *Effective Kan fibrations are closed under (functorial) push forward.*

2 Simplicial sets as a category with Moore structure

Theorem 3. *Consider the simplicial Moore path functor M from [BG12].*

(i) The functor M can be presented as a polynomial functor on an internal poset $\mathbb{T}_1 \to \mathbb{T}_0$. Here $\mathbb{T}_0$ is the simplicial set of traversals.

(ii) The simplicial Moore path functor is a symmetric Moore structure on simplicial sets.

The category of simplicial sets also admits a dominance of *effective cofibrations*. Therefore, there is a notion of effective Kan fibration in simplicial sets. A *Horn square* is a special type of mould square in simplicial sets:

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & s_i^* \left(\partial\Delta[n]\right) \\ \Big\downarrow & & \Big\downarrow \\ \Delta[n] & \xrightarrow[d_{i/i+1}]{} & \Delta[n+1] \end{array} \qquad (2)$$

Theorem 4. *A Horn square is a mould square. Therefore effective Kan fibrations are ordinary Kan fibrations. Moreover, there is an isomorphism of notions of fibred structure between*

- *being an effective Kan fibration;*
- *having a family of lifts against Horn squares which are stable under 'base change' along degeneracy maps.*

Therefore, effective Kan fibrations are a local notion of fibred structure. Lastly, a classical metatheory allows for an effective choice of fillers of ordinary Kan fibrations with respect to Horn squares.

The local structure of effective Kan fibrations gives a good prospect for addressing other challenges, such as a univalent universe and a model structure. Our approach from scratch can be compared to the recent work of Henry, Gambino, Henry, Sattler and Szumilo [GH19] [GSS19], which is more advanced, yet has to comply with the theoretical obstructions of ordinary Kan fibrations.

References

[BCP15] Marc Bezem, Thierry Coquand, and Erik Parmann. "Non-constructivity in Kan simplicial sets". In: *13th International Conference on Typed Lambda Calculi and Applications*. Vol. 38. LIPIcs. Leibniz Int. Proc. Inform. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2015, pp. 92–106.

[BG12] Benno van den Berg and Richard Garner. "Topological and simplicial models of identity types". In: *ACM Trans. Comput. Log.* 13.1 (2012), Art. 3, 44. ISSN: 1529-3785.

[BG16] John Bourke and Richard Garner. "Algebraic weak factorisation systems I: Accessible AWFS". In: *J. Pure Appl. Algebra* 220.1 (2016), pp. 108–147. ISSN: 0022-4049.

[GH19] Nicola Gambino and Simon Henry. "Towards a constructive simplicial model of univalent foundations". arXiv:1905.06281. 2019.

[GSS19] Nicola Gambino, Christian Sattler, and Karol Szumilo. "The constructive Kan-Quillen model structure: two new proofs". arXiv:1907.05394. 2019.