# Categorical structures for type theory in univalent foundations, II

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## 1 Introduction

Various categorical structures have been introduced for studying type theories. In the present project, continuing [2], we compare several of these structures, working in univalent foundations.

Specifically, we compare categories with families, relative universes, and several variants of these, and investigate how they interact with univalence/saturation and the Rezk completion.

More generally, we explore the differences and novelties of studying algebraic structures in univalent foundations, compared to in classical foundations.

All our results have been formalised in Coq, over the UniMath library; specifically, in the tagged version `2020-AKLV-HoTT-UF-abstract` of the `UniMath/TypeTheory` repository.

## 2 Comparing algebraic structures in the univalent setting

To meaningfully compare different kinds of structures in a classical setting, one must organise them into *categories*, and study *functors* between these categories. Equivalence of categories, for instance, gives a good notion of equivalence between two kinds of structures.

But the category structure is extra infrastructure that must be defined by hand. Off the shelf, the structures form just classes; and functions between these classes tell us little. Bijection between sets of structures, for instance, is not particularly meaningful or useful: it neither implies nor is implied by equivalence of the corresponding categories.

In the univalent setting, structures of some kind automatically form a *type*, which (thanks to its non-trivial equality types) carries much more information than the classical class of such structures. Typically, the type of widgets will correspond to the *groupoid core* of the category of widgets. (Precisely, this is univalence/saturation of the category of widgets.)

Equivalences of types of structures, or other properties of functions between these types, thus already give meaningful comparisons between the different kinds of structure.

In [2], we compared several different notions of structure at the level of types, giving functions between the types of such structures, and showing which of these functions are equivalences, embeddings, or surjections. In the present work, we raise these comparisons to the category level: we define (univalent) categories of these structures, and discuss how properties of functors between them correspond to the properties of the underlying functions.

# 3 Categorical structures for type theories

We mainly consider four types of structure: *categories with families*, *representable maps of presheaves*, *relative universes*, and *weak relative universes*.

A *category with families*, or *CwF* (Dybjer [4], as reformulated by Fiore [5] and Awodey [3]) consists of a category, whose objects are thought of as *contexts*, along with presheaves of *types* and *terms* connected by a map, and a *context extension* operation, characterised by a universal property. *Representable maps of presheaves* weaken this by just asserting existence of objects with the desired universal property, rather than an operation providing them.

*Relative universes* were introduced in [2]. They abstract away the role that presheaves play in the definition of CwF's: for a functor $J : \mathcal{C} \to \mathcal{D}$, a *$J$-relative universe* is a map in $\mathcal{D}$ together with an operation providing certain *$J$-pullbacks*. (For the Yoneda embedding $\mathsf{y}_{\mathcal{C}} : \mathcal{C} \to \mathsf{PreShv}(\mathcal{C})$, a $\mathsf{y}_{\mathcal{C}}$-relative universe is precisely a CwF structure on $\mathcal{C}$.) A *weak* $J$-relative universe is the same, but with just existence of suitable $J$-pullbacks, not a given operation.

# 4 Univalent categories and the Rezk completion

A notable feature of category theory in univalent foundations (introduced by Ahrens, Kapulkin, and Shulman [1]) is that many categories of interest are *univalent* (also called *saturated*): equality of their objects corresponds precisely to isomorphism. Classically, this can only hold in degenerate cases; but in the univalent setting, it holds for most naturally constructed categories.

Working in univalent categories has various payoffs: since "isomorphism is equality", for instance, objects specified by universal properties become literally unique, and so existence conditions often imply (unique) existence of operations picking witnesses.

If a category $\mathcal{C}$ is not univalent, this can be rectified by the *Rezk completion* construction, which performs a homotopy-quotient on the objects, replacing their original equality with the isomorphisms of $\mathcal{C}$, to give a new category $\mathsf{RC}(\mathcal{C})$, univalent and (weakly) equivalent to $\mathcal{C}$.

# 5 Summary

Our results are summarised by the following diagram of categories and functors:

$$
\begin{tikzcd}
\mathsf{SplTy}(\mathcal{C}) \arrow[r, leftrightarrow, "\simeq"] & \mathsf{Cwf}(\mathcal{C}) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "\mathrm{ff}"] & \mathsf{RelU}(\mathsf{y}_{\mathcal{C}}) \arrow[r, "\mathrm{ff}"] \arrow[d, "\mathrm{ff}"] & \mathsf{RelU}(\mathsf{y}_{\mathsf{RC}(\mathcal{C})}) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, leftrightarrow, "\simeq"] & \mathsf{Cwf}(\mathsf{RC}(\mathcal{C})) \arrow[d, leftrightarrow, "\simeq"] \\
& \mathsf{Rep}(\mathcal{C}) \arrow[r, leftrightarrow, "\simeq"] & \mathsf{RelWkU}(\mathsf{y}_{\mathcal{C}}) \arrow[r, leftrightarrow, "\simeq"] & \mathsf{RelWkU}(\mathsf{y}_{\mathsf{RC}(\mathcal{C})}) \arrow[r, leftrightarrow, "\simeq"] & \mathsf{Rep}(\mathsf{RC}(\mathcal{C}))
\end{tikzcd}
$$

Here, $\mathsf{SplTy}(\mathcal{C})$ is the category of split type-category structures on a base category $\mathcal{C}$; $\mathsf{Cwf}(\mathcal{C})$ the category of CwF structures on $\mathcal{C}$; $\mathsf{RelU}(F)$ (resp. $\mathsf{RelWkU}(J)$) the category of (weak) $J$-relative universes, for a functor $J$; and $\mathsf{Rep}(\mathcal{C})$ the category of representable maps of presheaves on $\mathcal{C}$.

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