∞-type theories

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∞-categorical structures and type-theoretic structures are (conjectured to be) closely related. Kapulkin and Szumiło [6] showed that ∞-categories with finite limits are equivalent to 1-categorical models of Martin-Löf type theory with dependent sum types and intensional identity types. Kapulkin [5] gave a way to construct pushforwards in ∞-categories with finite limits from dependent function types in type theories, but it remains open whether this construction is an equivalence.

A major problem in connecting ∞-categories with type theories is a difference in strictness of equalities. Most equalities in ∞-categories hold only up to homotopy, while some equalities in type theories like the substitution law hold on the nose. One approach to this problem is to replace an ∞-category by a strict model.

In this work we take another approach. Instead of making ∞-categories close to type theories, we make type theories close to ∞-categories. We consider a kind of type theory in which most equalities hold up to homotopy and call it an *∞-type theory*. Informally, an ∞-type theory is a higher-dimensional extension of a type theory with explicit substitution [3] and explicit conversion [4]. Equalities in an ∞-type theory are replaced by homotopies, which play a similar role to convertibility proof terms, but an ∞-type theory also has higher homotopies to speak about equalities between homotopies.

Because of its higher-dimensional nature, writing down the syntax of an ∞-type theory is as hard as writing down the algebraic definition of an ∞-category. Instead, we define the notion of an ∞-type theory abstractly, following the previous work of the second author [8]. In that work, the author proposed to identify type theories with categories equipped with certain structures. Basic results on type theories are then stated and proved in a purely categorical way. Therefore, many of the results in the previous work can be translated into the language of ∞-categories.

After developing some basic theory of ∞-type theories, we explore connections between ∞-categorical structures and ∞-type-theoretic structures. To begin with, we construct an ∞-type theory $\mathbb{E}_\infty$ such that theories written in $\mathbb{E}_\infty$ are equivalent to ∞-categories with finite limits. The ∞-type theory $\mathbb{E}_\infty$ is an ∞-type-theoretic analogue of Martin-Löf type theory with dependent sum types and *extensional* identity types. We note that, unlike extensional identity types in ordinary type theories, those in ∞-type theories do not destroy higher-dimensional structures, because extensionality in the ∞-type-theoretic context means that elements of an identity type $a = b$ are equivalent to homotopies between a and b. We then extend $\mathbb{E}_\infty$ to an ∞-type theory $\mathbb{E}^{\Pi}_{\infty}$ with dependent function types such that theories written in $\mathbb{E}^{\Pi}_{\infty}$ are equivalent to locally cartesian closed ∞-categories, which is an ∞-categorical analogue of the results of Clairambault and Dybjer [2] and Seely [7]. As another interesting example, we construct an ∞-type theory $\mathbb{R}_\infty$ such that theories written in $\mathbb{R}_\infty$ are equivalent to ∞-type theories. ∞-type theory does eat itself [1].

References

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