A Step towards Non-Presentable Models of Homotopy Type Theory

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One important aspect of homotopy type theory is the construction of *models*: $(\infty,1)$-categories in which we can interpret the axioms of our type theory. Studying various models can help us discern which statements can and cannot be proven with our given axiomatization.

An important first step towards constructing models was taken in [2] where the authors proved that the $(\infty,1)$-*category of Kan complexes* is a model for homotopy theory. In particular, their result also guarantees the soundness of homotopy type theory.

This result has been dramatically generalized by Shulman [7]. He proves that every *Grothendieck* $(\infty,1)$-*topos* is a model for homotopy type theory, by proving that such topos can be represented by a model structure, a *type-theoretic model topos*, which has all the properties we desire, and in particular strict univalent universes.

It was already established that every Grothendieck $(\infty,1)$-topos, as defined by Lurie [3], is equivalent to the simplicial nerve of a *model topos*: a model category that is Quillen equivalent to a left exact Bousfield localization of simplicial presheaves [6]. The insight of Shulman was to recognize that two processes of:

1. taking presheaf categories
2. taking left exact Bousfield localization

preserve models of homotopy type theory. Thus if we start with Kan complexes, which were already known to be a model of homotopy type theory, and apply these two steps we deduce that all Grothendieck $(\infty,1)$-toposes are a model of homotopy type theory.

Grothendieck $(\infty,1)$-toposes are in themselves fascinating objects of study, however, they share many properties which are not elementary. In particular, every Grothendieck $(\infty,1)$-topos ...

1. ... is presentable,
2. ... has small colimits,
3. ... as a particular implication of the previous point always has a standard natural number object.

As we do not expect all models of homotopy type to satisfy these non-elementary conditions, the next natural step is to direct our attention towards finding models of homotopy type theory where one (or all) of the conditions above fail.

In this work we take a first step towards showing the existence of non-presentable models of homotopy type theory, by constructing a non-presentable *elementary* $(\infty, 1)$-*topos* [4]. An elementary $(\infty, 1)$-topos shares many features with Grothendieck $(\infty, 1)$-toposes (such as descent, universes, natural number objects, ...), but is not required to be presentable and thus can include examples that are not Grothendieck $(\infty, 1)$-toposes. We will show that we can construct non-presentable elementary $(\infty, 1)$-toposes out of Grothendieck $(\infty, 1)$-toposes.

We will achieve this goal via the *filter construction*. The filter construction has been studied extensively for 1-categories. In particular, it is established that the filter construction applied to a Grothendieck topos with non-trivial filter will result in a non-presentable elementary topos [1].

In this talk we will prove that this result can be generalized to the $(\infty, 1)$-categorical setting. Concretely, we prove that for every elementary $(\infty, 1)$-topos $\mathcal{E}$ and filter of subobjects Φ, we can construct an elementary $(\infty, 1)$-topos $\prod_\Phi \mathcal{E}$, the *filter quotient*, which is in fact not presentable if the filter Φ is not principal (even if $\mathcal{E}$ itself was presentable) [5]. We will apply this result to the $(\infty, 1)$-category of Kan complexes to construct examples of elementary $(\infty, 1)$-toposes for which all three conditions above fail, meaning:

1. They are not presentable.
2. They do not have countable colimits.
3. The natural number object is non-standard.

How can we use this result to construct new and interesting models for homotopy type theory? By [2] we already know that Kan complexes are a model, so the main step is to prove that the filter construction preserves models of homotopy type theory. However, the filter construction is inherently a colimit construction (it is a filtered colimit of $(\infty, 1)$-categories) and unfortunately the results in [7] do not imply that a colimit of models of homotopy type theory is itself a model. Thus, in order to show filter quotients give us models for homotopy type theory the key step is to study colimits of such models.

References:

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