Investigations into syntactic iterated parametricity and cubical type theory

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We place ourselves in the context of syntactic interpretations of type theory which validate function or type extensionality axioms. Typical such translations are the setoid translation [BCH15, ABKT19], the groupoid translation [ST14], the half-way 1-truncated/2-truncated semi-simplicial translation [BCH15], the univalent parametricity translation [TTS18, TTS19], ... We are more precisely looking for syntactic translations which interpret parametric type theory [BCM15] and (variants of) cubical type theory [CCHM18].

We could have formulated in a type-theoretic language the standard cubical/parametricity presheaf models [BCM15, CCHM18] (which have been abundantly studied in a set-theoretic metalanguage). We consider instead a cubical/parametric syntactic model exposing the combinatorial and inherently dependent structure of the presheaf construction, similar in spirit to the dependently-typed presentation of semi-simplicial sets from a previous work [Her15]. Otherwise said, in light of Grothendieck's correspondence between fibred categories and indexed categories, we adopt an (iterated) "indexed" presentation of the face morphisms of such presheaves.

As target type theory, we take a relatively standard type theory with Σ-types, Π-types, a hierarchy of universe U_l, natural numbers and their finite intervals, a unit type with single inhabitant $\star$, coinductive streams with projections hd and tl, as well as a strict equality. The target could then typically be based on ETT.

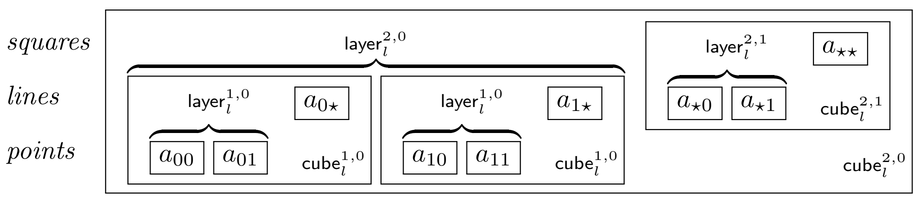
The basis of our dependently-typed construction is the following definition cubset_l of semi-cubical sets at some universe U_l, taken as the coinductive limit of truncated semi-cubical sets $\mathsf{cubset}_l^{<n}$, where by semi-cubical sets we mean cubical sets with only faces [BM17]:

$$
\begin{array}{lllll}
\mathsf{cubset}_l & & : & \mathsf{U}_{l+1} \\
\mathsf{cubset}_l & & \triangleq & \mathsf{cubset}_l^{\geq 0}(\star) \\[1ex]
\mathsf{cubset}_l^{\geq n} & (D : \mathsf{cubset}_l^{<n}) & : & \mathsf{U}_{l+1} \\
\mathsf{cubset}_l^{\geq n} & D & \triangleq & \Sigma R : \mathsf{cubset}_l^{=n}(D).\, \mathsf{cubset}_l^{\geq n+1}(D, R) \\[1ex]
\mathsf{cubset}_l^{<n} & & : & \mathsf{U}_{l+1} \\
\mathsf{cubset}_l^{<0} & & \triangleq & \mathsf{unit} \\
\mathsf{cubset}_l^{<n'+1} & & \triangleq & \Sigma D : \mathsf{cubset}_l^{<n'}.\, \mathsf{cubset}_l^{=n}(D)) \\[1ex]
\mathsf{cubset}_l^{=n} & (D : \mathsf{cubset}_l^{<n}) & : & \mathsf{U}_{l+1} \\
\mathsf{cubset}_l^{=n} & D & \triangleq & \mathsf{fullbox}_l^{n}(D) \to \mathsf{U}_l
\end{array}
$$

In the definition, $\mathsf{fullbox}_l^n(D) \triangleq \mathsf{box}_l^{n,n}(D)$ is defined by means of the following incremental characterisation of boxes (= border of cubes) and cubes (= filled boxes):

$$
\begin{array}{llll}
\mathsf{box}_l^{n,p,[p\leq n]} & (D : \mathsf{cubset}_l^{<n}) & : & \mathsf{U}_l \\
\mathsf{box}_l^{n,0} & D & \triangleq & \mathsf{unit} \\
\mathsf{box}_l^{n,p'+1} & D & \triangleq & \Sigma d : \mathsf{box}_l^{n,p'}(D).\, \mathsf{layer}_l^{n,p'}(D)(d) \\[1ex]
\mathsf{layer}_l^{n,p,[p<n]} & (D : \mathsf{cubset}_l^{<n})\ (d : \mathsf{box}_l^{n,p}(D)) & : & \mathsf{U}_l \\
\mathsf{layer}_l^{n,p} & D\ d & \triangleq & \begin{array}{l} \mathsf{cube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}_{l,L}^{n,p}(D)(d)) \\ \times\ \mathsf{cube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{hd}(D))(\mathsf{subbox}_{l,R}^{n,p}(D)(d)) \end{array} \\[1ex]
\mathsf{cube}_l^{n,p,[p\leq n]} & (D : \mathsf{cubset}_l^{<n})\ (E : \mathsf{cubset}_l^{=n}(D))\ (d : \mathsf{box}_l^{n,p}(D)) & : & \mathsf{U}_l \\
\mathsf{cube}_l^{n,p,[p=n]} & D\ E\ d & \triangleq & E(d) \\
\mathsf{cube}_l^{n,p,[p<n]} & D\ E\ d & \triangleq & \Sigma b : \mathsf{layer}_l^{n,p}(D)(d).\, \mathsf{cube}_l^{n,p+1}(D)(E)(d, b)
\end{array}
$$

which corresponds to the following organisation of the 3^n components of a n-cube (shown for $n = 2$), with box_l associating layers on the left and cube_l associating them on the right:



additionally, each atomic component at dimension n is a $\mathsf{cube}_l^{n,n}$

In the definition of $\mathsf{layer}_l^{n,p}$, $\mathsf{subbox}_{l,\epsilon}^{n,p}$ is a "face" operation again defined incrementally (not described here), and itself eventually requiring a coherence condition on the commutation of two $\mathsf{subbox}_{l,\epsilon}^{n,p}$ (for $\epsilon \in \{L, R\}$, representing the left and right p^{th} face of an n-box). It is to avoid requiring further coherence conditions in higher dimensions that we expect the equality of the target type theory to be strict.

Another key component of our construction is a similarly coinductive notion of dependent cubical sets $\mathsf{depcubset}_l : \mathsf{cubset}_l \to U_{l+1}$ with which we can define the dependent product and dependent sum of a cubical set A and of a dependent cubical set over A. It allows also to interpret U_l as the following stream of type cubset_{l+1}:

$$
\begin{array}{lcl}
\mathsf{univcubset}_l & : & \mathsf{cubset}_{l+1} \\
\mathsf{univcubset}_l & \triangleq & \mathsf{univcubset}_l^{\geq 0} \\
\\
\mathsf{univcubset}_l^{\geq n} & : & \mathsf{cubset}_{l+1}^{\geq n}(\mathsf{univcubset}_l^{<n}) \\
\mathsf{univcubset}_l^{\geq n} & \triangleq & (\mathsf{univcubset}_l^{=n}, \mathsf{univcubset}_l^{\geq n+1}) \\
\\
\mathsf{univcubset}_l^{<n} & : & \mathsf{cubset}_{l+1}^{<n} \\
\mathsf{univcubset}_l^{<0} & \triangleq & \star \\
\mathsf{univcubset}_l^{<n'+1} & \triangleq & (\mathsf{univcubset}_l^{<n'}, \mathsf{univcubset}_l^{=n'}) \\
\\
\mathsf{univcubset}_l^{=n} & : & \mathsf{cubset}_{l+1}^{=n}(\mathsf{univcubset}_l^{<n}) \\
\mathsf{univcubset}_l^{=n} & \triangleq & \lambda d : \mathsf{fullbox}_l^n(\mathsf{univcubset}_l^{<n}).\, \mathsf{depcubset}_l(\mathsf{univliftfullbox}_l^n(d))
\end{array}
$$

where $\mathsf{univliftfullbox}_l^n(d)$, defined mutually with $\mathsf{univcubset}_l^{<n}$, turns a n-box of cubical sets into a cubical set of n-boxes by distributing Σ-types over cubical sets.

We can then interpret a closed type A as a cubical set $[\![A]\!]$ and a closed term of closed type A as an inhabitant of $\mathsf{El}([\![A]\!])$ for $\mathsf{El}(D) \triangleq \mathsf{hd}(D)(\star)$, henceforth allowing to interpret a sequent $\Gamma \vdash t : A$ as a sequent $\vdash [\![\lambda\Gamma.t]\!]_{\Pi\Gamma.A} : \mathsf{El}([\![\Pi\Gamma.A]\!])$, though several details are yet to be completed at the time of writing this abstract.

The next step is to add reflexivities (degeneracies) to the cubical set structure, as well as considering cubes of cubical sets, so as to interpret abstraction and application over a variable of dimension, and, in particular to interpret parametric type theory [BCM15]. Adding contraction (cartesian structure) and exchange (symmetric group structure) should then be straightforward though tedious, because of the number of coherence conditions to explicitly consider. Moving from arbitrary relations to equivalences (using the symmetric definition of [AK18] which mechanically delivers Kan composition by construction) would eventually provide a syntactic model of univalent cubical type theory where univalence is expected to hold definitionally and where we also expect to get regularity for the universes and a definitional equality of functions (though presumably a non-standard one in the cartesian case).

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