A refinement of Gabriel-Ulmer duality

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Essentially algebraic theories and *generalized algebraic theories* provide an extension of the notion of algeraic theory which is fundamental to the semantics of type theory since the doctrines in which we interpret type theories – such as the (2-)category of *categories with families* – are themselves categories of models of essentially algebraic theories. This insight is sometimes summarized by the slogan 'type theory is algebraic'.

Cartmell's generalized algebraic theories [Car86] – which extend algebra by type dependency – and Freyd's essentially algebraic theories [Fre72] – which permit a controlled form of partiality – are commonly recognized as being equally expressive, and are both subsumed by the less syntactic and more abstract *finite-limit theories*, which are simply small finite-limit categories.

More specifically, for each generalized/essentially algebraic theory $\mathcal{T}$ there exists a small finite-limit category $\mathbb{C}$ such that

$$\mathrm{Mod}(\mathcal{T}) \simeq \mathbf{Lex}(\mathbb{C}, \mathbf{Set}),$$

i.e. the category of models of $\mathcal{T}$ is equivalent to the category of finite-limit-preserving ('lex') functors from $\mathbb{C}$ to the category of sets.

Classical Gabriel-Ulmer duality [GU71] states that the categories of models of such theories are precisely the *locally finitely presentable categories*, giving rise to a contravariant biequivalence

$$\mathbf{Lex} \simeq \mathbf{LFP}^{\mathsf{op}}$$

between the 2-categories of finite-limit categories and locally finitely presentable categories.

We propose a refinement of this duality based on the notion of *clan*, which was introduced by Taylor under the name 'category with display maps' [Tay87, 4.3.2], and later renamed by Joyal [Joy17, 1.1.1].

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Definition 1 A *clan* is a small category $\mathcal{T}$ with terminal object 1 equipped with a class $\mathcal{D}$ of *display maps* such that pullbacks of display maps along arbitrary maps exist and are again display maps, and display maps contain isomorphisms and terminal projections and are closed under composition.

A *model* of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \mathbf{Set}$ which preserves 1 and pullbacks of display maps. $\diamondsuit$

Clans refine finite-limit theories in that the 'same' finite-limit theory can be represented by different clans, thus we cannot expect to reconstruct a clan from its category of models alone.

To recover a duality we equip the category $\mathrm{Mod}(\mathcal{T})$ of models of $\mathcal{T}$ with additional information in form of a weak factorization system $(\mathcal{E}, \mathcal{F})$ which is cofibrantly generated by the morphisms of models induced by display maps[1].

Our main result is a way to recover $\mathcal{T}$ from $\mathrm{Mod}(\mathcal{T})$ and $(\mathcal{E}, \mathcal{F})$ up to Cauchy-completion, and a conjectured characterization of those locally finitely presentable categories equipped with weak factorization systems that arise as categories of models of clans.

Moreover we showcase a major advantage of clans over finite-limit theories, which is that different clanic representations of the same finite-limit theory when interpreted in higher types give rise to different 'homotopy-coherent generalizations' of the set-level algebraic structure. This way we can e.g. obtain a 'higher algebraic' description of a $(2,1)$-category of CwFs, which is relevant to non-strict initiality results.

References

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[1] The same w.f.s. construction was independently proposed by Simon Henry in a recent HoTTEST seminar.