Towards an enveloping ∞-topos of the effective topos

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Hyland's effective topos, $\mathcal{E}ff$ is a very well studied object within constructive mathematics and topos theory. From a logical point of view, it is a topos, and so has a rich internal language including all of higher order logic, and yet it satisfies all of the standard axioms of Russian constructive mathematics, including Church's thesis (all functions are computable), Markov's principle, and the axiom of countable choice. In addition it has been found to have a rich structure, studied by many people. An example of this is the lattice of Lawvere-Tierney topologies. In Hyland's original paper on the effective topos, he already showed that the poset of Turing degrees, an important structure in computability theory, can be embedded in the lattice of Lawvere-Tierney topologies in $\mathcal{E}ff$. Later Lee and Van Oosten showed this lattice has yet more structure, using a notion they called *sights*, combining computable functions with trees.

Meanwhile ∞-toposes are a generalisation of topos that have seen a lot of interest in recent years, notably with a lot of the general theory of Grothendieck ∞-toposes developed by Lurie. ∞-toposes build on the language of toposes by enriching the internal language further with many ideas that we are familiar with from homotopy type theory such as univalent universes and higher inductive types.[1] For Grothendieck toposes Lurie has shown that each topos can embedded in an *enveloping ∞-topos*. That is, an ∞-topos whose 0-truncated objects are precisely the original 1-topos, also satisfying a universal property that determines it uniquely up to equivalence. However, for elementary toposes, such as $\mathcal{E}ff$, the existence of enveloping ∞-toposes is still unclear. Potentially, enveloping ∞-toposes could allow one to study the original 1-topos through the enriched language of homotopy type theory. For example, any logical statement that holds for 0-truncated objects in the ∞-topos also holds for the original topos. The Lawvere-Tierney topologies of the original topos appear as *topological modalities* in the enveloping ∞-topos.

I will talk about the main difficulties in constructing an enveloping ∞-topos

[1] For Grothendieck ∞-toposes Shulman has now proved that one can always interpret homotopy type theory in the ∞-topos. For elementary ∞-toposes, a formal result remains open, but the idea of ∞-toposes as models of HoTT and HoTT as the internal language of ∞-toposes is still a useful intuitive picture.

for the effective topos, and some approaches that appear to be useful for solving them. In earlier joint work with Uemura, we showed that the most straightforward model of HoTT in cubical assemblies, based on ideas of Orton and Pitts, does *not* satisfy Church's thesis. It follows that the most natural definition of effective ∞-topos[2] is not enveloping for $\mathcal{E}ff$, since it does not satisfy a logical statement (Church's thesis) that holds in $\mathcal{E}ff$. However, for the case of Church's thesis, Uemura and I showed that one can produce a model of HoTT and Church's thesis, using Rijke, Shulman, and Spitters' notion of *topological modality*. I will discuss how this technique can be extended to produce a model structure on cubical assemblies that behaves closer to how the enveloping ∞-topos on $\mathcal{E}ff$ is expected to behave. This uses the notion of *left Bousfield localisation*, where one adds more trivial cofibrations to a model structure while keeping the class of cofibrations the same, in this case ensuring that the constant presheaf functor from assemblies to cubical assemblies preserves all covers. Time permitting, I will also mention an alternative approach based on a modification of the Henry-Gambino-Sattler-Szumiło model structure on simplicial sets.

[2]This ∞-topos will be described in a paper in preparation with Awodey, Frey and Anel.