THE HUREWICZ THEOREM IN HOMOTOPY TYPE THEORY

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Working in homotopy type theory as described in the book [6], we prove the **Hurewicz theorem**: for $n \geq 1$ and X a pointed, $(n-1)$-connected type, there is a natural isomorphism

$$\pi_n(X)^{ab} \cong H_n(X), \tag{1}$$

where on the left-hand-side we take the abelianization (which only matters when $n = 1$). Below we explain the ingredients that go into this, state a more general form of the Hurewicz theorem, and describe the results we obtain along the way. Before doing so, we give some motivation for the interest in this result.

In topology, homotopy groups are in a certain sense the strongest invariants of a topological space, and so their computation is an important tool when trying to classify spaces up to homotopy. In homotopy type theory, homotopy groups play a fundamental role in that they capture information about iterated identity types. Unfortunately, even in classical topology, the computation of homotopy groups is a notoriously difficult problem. Nevertheless, topologists have come up with a variety of powerful tools for attacking this problem, and one of the most basic tools is the Hurewicz theorem. In most cases, it is much easier to compute homology groups than homotopy groups, and so one can use the isomorphism from right to left to compute certain homotopy groups. Moreover, one can apply the theorem even when X is not $(n-1)$-connected using the following technique. Let $X\langle n-1\rangle$ denote the fibre of the truncation map $X \to \left\|X\right\|_{n-1}$ over the image of the basepoint. Then $X\langle n-1\rangle$ is $(n-1)$-connected and $\pi_n(X\langle n-1\rangle) \cong \pi_n(X)$, so $\pi_n(X)^{ab} \cong H_n(X\langle n-1\rangle)$. The Serre spectral sequence [4] can often be used to compute the required homology group.

We now give a more detailed account of our work, starting with the hypothesis for the theorem: a type X is $\boldsymbol{(n-1)}$**-connected** if its $(n-1)$-truncation is contractible.

Moving on to the left-hand-side, given a pointed type X and $n \geq 1$, the **homotopy group** $\pi_n(X)$ is defined to be the set-truncation $\left\|\Omega^n(X)\right\|_0$ of the iterated loop space. This carries a natural group structure, which is abelian when $n \geq 2$. (Throughout, when we use the word **group**, we mean a set (0-truncated type) with a binary operation satisfying the usual properties.) An **abelianization** of a group G is a group homomorphism $G \to G^{ab}$ to an abelian group which is initial among such homomorphisms. It is straightforward to show that abelianizations exist, although the most direct method is somewhat tedious. We give a more efficient construction as a higher inductive type with one point constructor $\eta : G \to G^{ab}$, one 1-path constructor asserting that $\eta(a \cdot (b \cdot c)) = \eta(a \cdot (c \cdot b))$ for $a, b, c : G$, and one 2-path constructor enforcing that G^{ab} is a set. It is immediate that the type of abelianizations of G is contractible and that the identity map serves as the abelianization of an abelian group.

Now we explain the homology groups that appear on the right-hand-side of the Hurewicz isomorphism. First, recall that given two pointed types X and Y, the **smash product** $X \wedge Y$ is defined to be the higher inductive type with constructors:

- $\mathsf{sm} : X \times Y \to X \wedge Y$.
- $\mathsf{auxl} : X \wedge Y$.
- $\mathsf{auxr} : X \wedge Y$.
- $\mathsf{gluel} : \prod_{(y:Y)} \mathsf{sm}(x_0, y) = \mathsf{auxl}$.
- $\mathsf{gluer} : \prod_{(x:X)} \mathsf{sm}(x, y_0) = \mathsf{auxr}$.

Next, given an abelian group A and $m \geq 1$, [5] constructed an **Eilenberg-Mac Lane space** $K(A, m)$, which is an m-truncated, $(m-1)$-connected, pointed type with a canonical isomorphism $\pi_m(K(A, m)) \cong A$. Using that $\Omega(K(A, m+1)) \simeq K(A, m)$, one can construct for any pointed type X and any $n \geq 1$ a sequential diagram

$$\pi_{n+1}(X \wedge K(A,1)) \longrightarrow \pi_{n+2}(X \wedge K(A,2)) \longrightarrow \pi_{n+3}(X \wedge K(A,3)) \longrightarrow \cdots \tag{2}$$

The ***n*th homology group** $H_n(X; A)$ of X with coefficients in A is defined to be the colimit of this sequence. We write $H_n(X)$ for $H_n(X; \mathbb{Z})$, where $\mathbb{Z}$ is the group of integers. We have now explained everything that appears in the isomorphism (1).

To state the more general version of the Hurewicz theorem, we need to introduce one more ingredient that appears on the left-hand-side. Given abelian groups $A, B : \mathsf{Ab}$, a **tensor product** of A and B consists of an abelian group T together with a map $t : A \to_{\mathsf{Grp}} B \to_{\mathsf{Grp}} T$ such that, for any abelian group C, the map

$$t^* : (T \to_{\mathsf{Grp}} C) \longrightarrow (A \to_{\mathsf{Grp}} B \to_{\mathsf{Grp}} C)$$

given by composition with t is an equivalence. When such a map t exists, we write $A \otimes B \cong T$.

The generalized **Hurewicz theorem** says that for X a pointed, $(n-1)$-connected type and A an abelian group, there is a natural isomorphism

$$\phi : \pi_n(X)^{ab} \otimes A \cong H_n(X; A).$$

In order to define the natural map ϕ, we define and study a more general natural map

$$\mathsf{smashing} : (X \to_{\bullet} Y \to_{\bullet} Z) \longrightarrow (\pi_n(X) \to_{\mathsf{Grp}} \pi_m(Y) \to_{\mathsf{Grp}} \pi_{n+m}(Z))$$

for any pointed types X, Y and Z and any $n, m \geq 1$. While this map lands in group homomorphisms between (0-truncated) groups, in order to construct it, we pass through *magmas*. A **magma** is a type M with a binary operation $\cdot : M \times M \to M$, with no conditions. As a technical trick which simplifies the formalization, we work with *weak magma morphisms*. A **weak magma morphism** from a magma M to a magma N is a map $f : M \to N$ which *merely* has the property that it respects the operations. This is sufficient for our purposes, because when M and N are groups, it reproduces the notion of group homomorphism. All loop spaces are magmas under path concatenation, and many natural maps involving loop spaces are weak magma morphisms. By working with magmas, we can factor the map $\mathsf{smashing}$ into simpler pieces, and still land in group homomorphisms at the end, without keeping track of higher coherences.

As a key step towards the Hurewicz theorem, we prove that if X is a pointed, $(n-1)$-connected type ($n \geq 1$) and Y is a pointed, $(m-1)$-connected type ($m \geq 1$), then $X \wedge Y$ is $(n+m-1)$-connected and $\pi_{n+m}(X \wedge Y)$ is the tensor product of $\pi_n(X)^{ab}$ and $\pi_m(Y)^{ab}$ in a natural way. The tensor product structure comes from the map $\mathsf{smashing}$ applied to the natural map $X \to_{\bullet} Y \to_{\bullet} X \wedge Y$. Taking Y to be $K(A, m)$ in this result shows that the groups appearing in the sequential diagram (2) are tensor products of $\pi_n(X)$ and A. The proof of the Hurewicz theorem follows from showing that the induced maps are isomorphisms.

In order to work with natural transformations, we use the framework of wild 1-categories, and make use of the Yoneda lemma in this setting. We also rely on work of [2] and [3].

Formalization of these results is in progress, using the Coq HoTT library [1].

References

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