Non-wellfounded sets in HoTT

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In this talk, I wish to present a non-wellfounded model of set theory in Homotopy Type Theory (HoTT) which satisfies Aczel's Anti-Foundation Axiom, and where the equality is the identity type. This work is, in collaboration with Elisabeth Bonnevier, being formalised in Agda, building on the HoTT-Agda library. The current source code can be found at `https://git.app.uib.no/hott/hott-set-theory`.

Non-wellfounded set theory Most set-theoretical axioms tell us how to construct new sets: pairs, unions, subsets, and exponential- or powersets. **The axiom of foundation**, on the other hand, puts restrictions on which sets are allowed to exist: *There are no non-wellfounded sets.* This restriction allows us to prove things like: the set of natural numbers is the only set where an element is either zero or a successor of another element in the set (i.e. of the form $x \cup \{x\}$, in the most common encoding). But even if one abandoned the axiom of foundation, all one have to do to re-validate the statement is to add the qualifier "wellfounded" to the theorems and they will hold true.

If one were to go this way, and exclude the axiom of foundation, the immediate question is: *What are the non-wellfounded sets?* One possible answer is provided by **Aczel's Anti-Foundation Axiom** (AFA), which was first introduced by Forti and Honsell[3]. It states given any graph, expressed as a set, one can find a unique assignment of sets to the nodes, such that an edge in the graph corresponds to the elementhood relation between the assigned sets.

For instance, the graph with a single node and a single edge would be assigned the unique set $x = \{x\}$. In fact, this particular is a fixed point of the successor operation since $x \cup \{x\} = x \cup x = x$, which means that one could take $\mathbb{N} \cup \{x\}$, and use as a counter-example to the previous theorem which was provable by foundation. Figure 1 gives another example of a set defined using AFA.

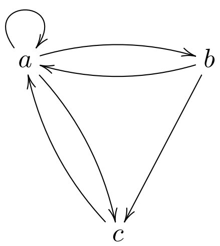

Figure 1: The Anti-Foundation Axiom states that for every graph there is a unique assignment of sets to the nodes such that the elementhood relation coincides with the edge-relation. The graph above would define the sets: $a = \{a, b, c\}$, $b = \{a, c\}$ and $c = \{a\}$.

Models in type theory Models of AFA in dependent type theory go back to a paper by Lindström[5] from 1989, which uses M-types to construct a setoid model of CZF^- + AFA (CZF without foundation and with Anti-Founation). This is a natural, dual construction of Aczel's original model of CZF in Martin Löf's type theory[1].

In HoTT, the most natural notion of model is one where the equality is the identity type. So far, all the models of this kind have satisfied the Foundation. This leaves us the task of constructing a model of set theory, where the identity type is the equality, and which satisfies AFA along with other set theoretical axioms.

Central to this endeavour is the powerset operation, $P_U\,X := \sum_{A:U}(A \hookrightarrow X)$, which maps a type X to the mere set of embeddings of small types into X. In fact, the rest of the work can then be summarised in terms of P_U coalgebra as follows:

2. For a (locally U-small) terminal coalgebra for P, the fixed point model constructed in point 1 also satisfies Anti-Foundation.
3. Type V^0_∞, described below, is indeed a locally U-small, terminal coalgebra for P.

This nicely complements the construction in a previous article[4], which, from this point constructs a initial algebra for P_U, and shows that it satisfies foundation.

Technical details The model presented in this talk is constructed from the M-type, $M_\infty := M_{A:U}A$, for a given univalent universe U. M-types in general was shown to be constructible in HoTT by Ahrens et al.[2]. Elements in this particular M-type can be seen as pairs (X, f), where $X : U$ and $f : X \to M_\infty$. By taking the image of the co-iterative endo-function, $M_\infty \to M_\infty$, which forces, in each element, the function f to become injective, one obtains a quotient (and sub-) type, which will be called V^0_∞ which will be our model. Figure 2 shows the coalgebra homomorphism from the universal property of the M-type and the image factorisation which defines our V^0_∞.

$$\begin{tikzcd}
\sum_{A:U} A \to M_\infty \arrow[rr, "\sum_A(\text{corec-M}_\infty\,(\sigma\circ\text{desup}))^A"] & & \sum_{A:U} A \to M_\infty \\
\sum_{A:U} A \to M_\infty \arrow[u, "(A,f)\mapsto(\text{Image}(f),\text{image-inclusion}(f))", "\sigma"'] & & \\
M_\infty \arrow[u, "\text{desup}"] \arrow[rr, "\text{corec-M}_\infty\,(\sigma\circ\text{desup})"] \arrow[dr, two heads] & & M_\infty \arrow[uu, "\text{desup}"'] \\
& V^0_\infty \arrow[ur, hook] &
\end{tikzcd}$$

Figure 2: Diagram defining V^0_∞ from the universal property of M_∞ as a terminal coalgebra and image factorisation.

References

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[1] A locally U-small type is a type X such that for all $x, x' : X$ the identity type $x = x'$ is equivalent to a type in U.