Formalising the Escardó-Simpson Closed Interval Axiomatisation in Univalent Type Theory

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The classical specification of the real numbers is as the unique complete Archimidean field. Here we instead explore an axiomatization of '*interval objects*' by Escardó and Simpson based on midpoint algebras [3]. The completeness axiom is replaced by an iteration property, and the closed interval is specified by a universal property that gives a recursion principle for real numbers. This axiomatisation supports constructive mathematics by design, making it attractive to theory of computation, wherein interval objects can be viewed as abstract data types for real numbers [4]. It is also of particular interest to the HoTT/UF community; indeed, in the HoTT book it is conjectured that the defined higher-inductive type for Cauchy reals "probably coincide with the Escardó-Simpson reals"[1], a presumption that was proved by Booij recently in his Ph.D. thesis [2]. The HoTT book also recognises another interesting property of these reals: that they "can be stated in any category with finite products", allowing for a general notion of interval objects which often coincide with previous concepts. In particular, it has been shown that in **Set** $[-1,1]$ is an interval object, and in **Top** $[-1,1]$ with the expected Euclidean topology is an interval object. We now give a work-in-progress formulation of this work within univalent type theory.

In this talk, we will outline the concepts and theorems proposed by Escardó and Simpson, and show how they have been newly formalised in AGDA. Furthermore, we will discuss the formulation of this axiomatisation in the language of univalent type theory – in particular, we characterise equality on interval objects and related structures by applying a structure identity principle. The formalisation is implemented within Escardó's AGDA library TYPETOPOLOGY[2], based on univalent type theory. In this abstract, we employ the type theory, notation and terminology used in the HoTT book.

Bipointed convex bodies. Our interval objects are conceived to represent any line segment – a bounded, convex subset of the real line. These sets are *convex* because they contain every point on the line between its endpoints. In order to define convexity, we use the idea of taking the midpoint between two numbers, and furthermore that taking such a midpoint can be infinitely iterated [4].

We start with the structure of a *midpoint algebra*, which is a *magma* (a type equipped with a binary operation; $\mathsf{Magma} :\equiv \sum_{(A:\mathcal{U})}(A \to A \to A)$) where the type A is an h-set and the operation ($\oplus$, called the *midpoint operator*) is idempotent, commutative and transpositional. These properties correspond to types $\mathsf{Magma} \to \mathcal{U}$, e.g. $\mathsf{transpositional}(A,\oplus) :\equiv \prod_{(a,b,c,d:A)}((a\oplus b)\oplus(c\oplus d) = (a\oplus c)\oplus(b\oplus d))$. We then add two further properties to this structure: *cancellation* and *iteration*.

The cancellation property says that if $a \oplus c = b \oplus c$ then $a = b$; adding this gives us a *cancellative midpoint algebra*. The set $\mathbb{R}^n$ is a cancellative midpoint algebra closed under the binary midpoint function $\lambda xy.\frac{1}{2}(x+y)$; as are various subsets of $\mathbb{R}^n$, such as the rationals. Furthermore, given two rational endpoints, e.g. -1 and 1, we could use the midpoint function to generate any rational number in $[-1,1]$ – but we cannot generate *any* particular point on the convex line. For this, we require our version of the completeness axiom.

The iteration property states that there is an operator $\mathrm{M} : A^{\mathbb{N}} \to A$ that gives the 'infinitely iterated' midpoint of a stream of points of A. Formally, this operator is defined by two sub-properties:

$$\mathsf{iterative}(A,\oplus) :\equiv$$

$$\sum_{(\mathrm{M}:A^{\mathbb{N}}\to A)}\Bigg(\Big(\prod_{(a:A^{\mathbb{N}})}\mathrm{M}\,a = \mathrm{head}\,a\oplus\mathrm{M}(\mathrm{tail}\,a)\Big)\times\Big(\prod_{(x,a:A^{\mathbb{N}})}\Big(\prod_{(i:\mathbb{N})} ai = xi\oplus a(i+1)\Big)\to \mathrm{head}\ a = \mathrm{M}\,x\Big)\Bigg).$$

[1] Page 538, Notes on Chapter 11 in [6].
[2] https://www.cs.bham.ac.uk/~mhe/agda-new/Escardo-Simpson-LICS2001.html

The first sub-property characterises the M operator, while the second gives a computation rule for it with respect to a second stream which corresponds to the iteration on the first. From these sub-properties, we can prove the M operator satisfies (i) $\mathrm{M}(\lambda - .x) = x$, (ii) $\mathrm{M}(\lambda i.\,\mathrm{M}(\lambda j.xij)) = \mathrm{M}(\lambda i.\,\mathrm{M}(\lambda j.xji))$ and (iii) $\mathrm{M}(\lambda i.xi \oplus yi) = \mathrm{M}\,x \oplus \mathrm{M}\,y$.

Adding iteration to a cancellative midpoint algebra gives us the structure we call an *abstract convex body*. Every line segment of $\mathbb{R}^n$ is an abstract convex body; following this fashion, a closed line segment corresponds to a *bipointed convex body* [3]:

$$\textsf{Bi-convex-body} :\equiv \sum_{((A,\oplus):\textsf{Magma})} \left(\left(\begin{array}{c} \textsf{is-set } A \times \textsf{idempotent}(A,\oplus) \times \textsf{commutative}(A,\oplus) \\ \times\ \textsf{transpositional}(A,\oplus) \times \textsf{cancellative}(A,\oplus) \times \textsf{iterative}(A,\oplus) \end{array} \right) \times A \times A \right).$$

In our formulation, a closed and bounded line segment – called an *interval object* – on a given type A is defined as a bipointed convex body with underlying type A that satisfies the following universal property.

Universal property for interval objects. Given two bipointed convex bodies $(A, \oplus_A, \textsf{props}_A, u, v)$ and $(B, \oplus_B, \textsf{props}_B, s, t)$, a map $f : A \to B$ is a *midpoint homomorphism* if $f(x \oplus_A y) = f(x) \oplus_B f(y)$. We have already seen that M is a midpoint homomorphism (further, every midpoint homomorphism is automatically an M homomorphism). The universal property that characterises interval objects states that given two interval objects there is a *unique* midpoint homomorphism $h : A \to B$ which preserves the bipointed structure. Therefore, a given bipointed convex body is an interval object if the following type is inhabited:

$$\textsf{is-interval-object } (A, \oplus_A, \textsf{props}_A, u, v) :\equiv$$
$$\prod_{((B,\oplus_B,\textsf{props}_B,s,t):\textsf{Bi-convex-body})} \textsf{is-singleton}\Big(\sum_{(h:A\to B)} \Big((hu = s) \times (hv = t) \times \prod_{(x,y:A)} (h(x \oplus_A y) = hx \oplus_B hy) \Big) \Big).$$

The uniqueness of this map amounts to the requirement that the sigma type is a singleton. Thus, given an interval object as above, there is a unique map $\textsf{affine}_A : A \to A \to A \to A$ where $\textsf{affine}_A(a, b)$ is defined as the unique map h of the universal property on the interval objects $(A, \oplus_A, \textsf{props}_A, u, v)$ and $(A, \oplus_A, \textsf{props}_A, a, b)$. This affine map transforms a point $x : A$ on the interval object with endpoints u, v into the relative point $\textsf{affine}_A(a, b, x) : A$ on the interval object with the same underlying convex body, but with endpoints a, b.

Deriving operations and properties from the axioms. We now fix the interval object $(\mathbb{I}, \oplus, \textsf{props}, -1, +1)$ where $\mathbb{I}$ is an h-set representing the closed and bounded real interval $[-1, 1]$, with $-1, +1 : \mathbb{I}$ representing the endpoints. The term $-1 \oplus +1 : \mathbb{I}$ clearly represents the number 0, and all other numbers in the interval can be represented by terms of $\mathbb{I}$ iteratively generated from these endpoints by $\oplus : \mathbb{I} \to \mathbb{I} \to \mathbb{I}$ and $\mathrm{M} : \mathbb{I}^{\mathbb{N}} \to \mathbb{I}$. The universal property gives us the unique map $\textsf{affine}(a, b) : \mathbb{I} \to \mathbb{I} \to \mathbb{I} \to \mathbb{I}$ for any $a, b, : \mathbb{I}$, which transforms a representation of a point in $[-1, 1]$ to a representation of a point in the sub-interval with endpoints represented by a and b.

The negation operator can be defined as $\textsf{neg}(x) :\equiv \textsf{affine}(+1, -1, x)$, which is a midpoint homomorphism satsifying $\textsf{neg}(-1) = +1$ and $\textsf{neg}(+1) = -1$. From the uniqueness of $\textsf{affine}$ and the fact that the composition of any two midpoint homomorphisms is a midpoint homomorphism, it can be proved that for all $x : \mathbb{I}$, $\textsf{neg}(\textsf{neg}(x)) = x$. The multiplication operator is defined as $\textsf{mul}(x, y) :\equiv \textsf{affine}(\textsf{neg}(x), x, y)$; commutativity and associtativity are again formalised using the uniqueness of $\textsf{affine}$. We can even define (medial) power series using the M operator. The fact that these operations and properties are derived from the axioms, rather than axioms themselves, highlights the conciseness of our approach.

Of course, as we are working in a closed and bounded interval, we cannot define addition. However, by adding a single extra axiom, we can define truncated addition and subtraction, as well as operators for maximum, minimum and absolute value [4]. This axiom is the assumption of a function $\textsf{double} : \mathbb{I} \to \mathbb{I}$ which performs a truncated doubling of a term in the interval object. From this, for example, truncated addition and subtraction can be defined as $x +_{\mathbb{I}} y = \textsf{double}(x \oplus y)$ and $x -_{\mathbb{I}} y = \textsf{double}(x \oplus \textsf{neg}(y))$, respectively.

A note on our type theory. The axiomatisation presented thus far can be implemented within plain dependent type theory, except that the proof M satisfies property (ii) invokes function extensionality. However, we use the concepts of univalent mathematics such as h-sets and contractibility throughout. Overleaf, we introduce the structure identity principle for interval objects, which utilises the univalence axiom.

Structure identity principle. The structure identity principle is described in the HoTT book as "an informal principle that expresses that isomorphic structures are identical" [6]. The univalence axiom gives this principle for general types with no additional structure by stating that all type universes are *univalent* (i.e. for all types A, B, the canonical map $\mathsf{id\text{-}to\text{-}equiv}\ A\ B : (A = B) \to (A \simeq B)$ is an equivalence).

$$\mathsf{is\text{-}equiv} f :\equiv \Big(\textstyle\sum_{(g:B\to A)} (\prod_{(a:A)} g(fa) = a) \Big) \times \Big(\textstyle\sum_{(h:B\to A)} (\prod_{(b:B)} f(hb) = b) \Big)$$

$$A \simeq B :\equiv \textstyle\sum_{(f:A\to B)} (\mathsf{is\text{-}equiv} f)$$

$$\mathsf{is\text{-}univalent}\ \mathcal{U} :\equiv \textstyle\prod_{(A,B:\mathcal{U})} \Big(\mathsf{is\text{-}equiv}\ (\mathsf{id\text{-}to\text{-}equiv}\ A\ B) \Big)$$

Thus, univalence characterises equality by equivalence for general types, i.e. $(A = B) \simeq (A \simeq B)$. The structure identity principle characterises equality by a notion of equivalence for types $\Sigma S :\equiv \sum_{(A:\mathcal{U})} S$ specified by a structure $S : \mathcal{U} \to \mathcal{V}$. For example, the specification of a Magma is $S_{\mathsf{Magma}} :\equiv A \to A \to A$.

There are several structure identity principles in the literature [6] [5] [1]. It is convenient for our purposes to use [5], which is already implemented within TypeTopology[3]. In this setting, equality is characterised using a general theorem for a *standard notion of structure* (SNS). Therefore, for a given structure ΣS, a characterisation of equality is derived immediately by constructing an SNS. An SNS for type universes $\mathcal{U}, \mathcal{V}, \mathcal{W}$ is a sigma type consisting of:

- A structure specification $S : \mathcal{U} \to \mathcal{V}$.
- A map of homomorphisms, $\iota : \prod_{((A,s),(B,t):\Sigma S)} ((A \simeq B) \to \mathcal{W})$.
- A proof the identity equivalence gives a homomorphism $\rho : \prod_{((A,s):\Sigma S)} (\iota\ (A,s)\ (A,s)\ (\mathsf{id\text{-}to\text{-}equiv}\ A\ A))$.
- A proof that, given $A : \mathcal{U}$ and $s, t : SA$, the canonical map $\theta : (s = t) \to (\iota\ (A,s)\ (A,t)\ (\mathsf{id\text{-}to\text{-}equiv}\ A\ A))$ defined $\theta\ (\mathit{refl}\ s) :\equiv\ \rho(A, s)$ is an equivalence.

Given any $(S, \iota, \rho, \theta) : \mathsf{SNS}$, we define the type of structural equivalences for the type ΣS using ι :

$$(A, s) \simeq_{(S,\iota,\rho,\theta)} (B, t) :\equiv \sum_{(f:A\to B)} \sum_{(i:\mathsf{is\text{-}equiv} f)} (\iota\ (A, s)(B, t)\ i).$$

The following general theorem can then be applied for any structure with an SNS in order to characterise equality on that structure by the above notion of equivalence:

$$\mathsf{characterization\text{-}of\text{-}{=}} : \prod_{((S,\iota,\rho,\theta):\mathsf{SNS})} \prod_{(A,B:\Sigma S)} \Big((A = B) \simeq (A \simeq_{(S,\iota,\rho,\theta)} B) \Big)$$

We apply the above structure identity principle to get a characterisation of equality for midpoint algebras, convex bodies and interval objects *relative to a given universe*. A bipointed convex body with underlying type $A : \mathcal{U}$ is an interval object *relative to a given universe* $\mathcal{V}$ if, given any bipointed convex body with underlying type $B : \mathcal{V}$, there exists a unique midpoint homomorphism $h : A \to B$ that preserves the bipointed structure. (The universal property of interval objects states that a bipointed convex body is an interval object if it is an interval object *relative to any universe* $\mathcal{V}$.)

For interval objects in universe $\mathcal{U}$ *relative to a given universe* $\mathcal{V}$, the structure specification $S_{\mathsf{int\text{-}obj}}$ is that of a bipointed convex body (given on previous page). We then define the map of homomorphisms as expected: they are those midpoint homomorphisms that preserve the bipointed structure of the two convex bodies:

$$\iota_{\mathsf{int\text{-}obj}}\ (A, \oplus_A, \mathsf{props}_A, u, v)\ (B, \oplus_B, \mathsf{props}_B, s, t)\ (f, i) :\equiv \Big(\prod_{(x,y:A)} (f(x \oplus_A y) = fx \oplus_B fy) \Big) \times \begin{pmatrix} (fu = s) \\ \times (fv = t) \end{pmatrix}.$$

Constructing the terms $\rho_{\mathsf{int\text{-}obj}}$ and $\theta_{\mathsf{int\text{-}obj}}$ completes the SNS. The general theorem is then applied to achieve a characterisation of equality on interval objects. Finally, we can show that two interval objects in the same universe are equivalent by the above definition and, thus, are identical.

We have formalised this characterisation of equality for midpoint algebras, convex bodies and interval objects in Agda [4].

[3]https://www.cs.bham.ac.uk/~mhe/agda-new/UF-SIP.html
[4]https://www.cs.bham.ac.uk/~mhe/agda-new/UF-SIP-IntervalObject.html

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