# Domain theory in predicative Univalent Foundations

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We report on the development of a constructive and predicative account of domain theory in Univalent Foundations. By *predicative* we mean that we work in the absence of propositional resizing axioms. We show that various order-theoretic statements (e.g. Zorn's Lemma) imply particular propositional resizing axioms. Our aim is not only to obtain a constructive and predicative account of domain theory, but to enhance our understanding of (the ramifications of) impredicativity in Univalent Foundations.

**Impredicativity.** Resizing rules for propositions in Univalent Foundations were introduced by Vladimir Voevodsky [10]. Instead of adding structural rules to the type theory, we consider resizing *axioms* [9, Section 3.5]. We do so for two reasons. Firstly, the axioms follow from excluded middle and are therefore known to be consistent by the simplicial sets model [5]. (However, the axioms have not been given a constructive interpretation, so far.) Secondly, studying axioms rather than rules allows to prove things about the axioms in the theory, rather than the meta-theory.

The presence of such resizing axioms is said to make the type theory *impredicative*. An example of the use of such axioms is that the powerset is well-behaved in impredicative type theory: for instance, it has arbitrary unions if and only if a particular resizing axiom holds [4, `existence-of-unions-gives-PR`]. In the absence of such resizing axioms, we say that the type theory is *predicative*. We will elaborate on two resizing axioms now.

**Definition 1.** Let $\mathcal{U}$ and $\mathcal{V}$ be universes and let $X : \mathcal{U}$. We say that $X$ *has size* $\mathcal{V}$ if we have a type $Y : \mathcal{V}$ and an equivalence $Y \simeq X$, i.e. $X$ $\textsf{has-size}$ $\mathcal{V} :\equiv \sum_{Y:\mathcal{V}} Y \simeq X$.

**Resizing axiom** ($\Omega_{\mathcal{U}}$-impredicativity)**.** Let $\mathcal{U}$ be a type universe. The resizing axiom $\Omega_{\mathcal{U}}$*-impredicativity* asserts that the type $\Omega_{\mathcal{U}} :\equiv \sum_{P:\mathcal{U}} \textsf{is-prop}(P)$ has size $\mathcal{U}$.

**Resizing axiom** (Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$)**.** Let $\mathcal{U}$ and $\mathcal{V}$ be universes. The resizing axiom *Propositional-Resizing*$_{\mathcal{U},\mathcal{V}}$ asserts that every proposition in $\mathcal{U}$ has size $\mathcal{V}$.

**Domain theory.** Domain theory [1] is a branch of order-theory with many applications in the semantics of programming languages [8] and topology [7]. Its basic objects are *directed complete posets* (*dcpos*): posets that are required to have so-called directed joins. Often, we also wish to consider *pointed dcpos*: dcpos that moreover have a least element. We say that a pointed dcpo is *non-trivial* if it has an element other than the least element.

**Impredicativity in domain theory.** Traditionally, dcpos are defined using powersets. Instead, our predicative version defines $\mathcal{V}$-dcpos: posets that have joins for directed *families* indexed by types in some fixed universe $\mathcal{V}$. We think of $\mathcal{V}$ as a universe of "small" types. Accordingly, we make the following definition.

**Definition 2.** A $\mathcal{V}$-dcpo $(D, \sqsubseteq)$ is *small* if its carrier $D$ has size $\mathcal{V}$.

It should be noted that, predicatively, posets can already be quite involved, because, like categories in traditional settings, they may fail to be (locally) small. Indeed, after developing some theory of size, we can prove the following.

**Theorem 1.** *The existence of a non-trivial small pointed dcpo is equivalent to impredicativity for small $\neg\neg$-stable propositions.*

*More precisely, if we have a small non-trival pointed $\mathcal{V}$-dcpo $\mathcal{D}$, then the type $(\Omega_{\mathcal{V}})_{\neg\neg} :\equiv \sum_{P:\mathcal{V}} \mathsf{is\text{-}prop}(P) \times (\neg\neg P \to P)$ of $\neg\neg$-stable propositions in $\mathcal{V}$ has size $\mathcal{V}$.*

*The converse is established by the fact that $(\Omega_{\mathcal{V}})_{\neg\neg}$ is a non-trivial pointed $\mathcal{V}$-dcpo.*

One could argue that the theorem above is not very surprising, because the notion of a small dcpo refers directly to size. What is perhaps more surprising, is that we can relate impredicativity to purely order-theoretic statements, as illustrated by the following two theorems.

We study (a variation of) Zorn's Lemma, which says that every pointed dcpo must have a maximal element. We consider a version parameterised by universes here.

**Definition 3.** Let $\mathcal{V}$, $\mathcal{U}$ and $\mathcal{T}$ be universes. *Zorn's-Lemma*$_{\mathcal{V},\mathcal{U},\mathcal{T}}$ says that every pointed $\mathcal{V}$-dcpo $(D, \sqsubseteq)$ with $D : \mathcal{U}$ and $\sqsubseteq$ taking values in $\mathcal{T}$, has a maximal element.

By constructing a particular pointed dcpo, we can show the following.

**Theorem 2.** Zorn's-Lemma$_{\mathcal{V},\mathcal{V}^+ \sqcup \mathcal{U},\mathcal{V}}$ *implies* Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$.

It is worth mentioning that Zorn's Lemma does not imply excluded middle nor the axiom of choice (in the absence of excluded middle) [2].

A highlight of domain theory is *Pataraia's fixed point theorem* [6, 3], which says that every monotone endofunction on a pointed dcpo has a least fixed point. The theorem may be seen as a topos-valid (i.e. constructive, but impredicative) alternative to the classical Bourbaki-Witt fixed point theorem. Pataraia proved the theorem by considering inflationary endomaps: a monotone endomap $f$ on $X$ is *inflationary* if $x \sqsubseteq f(x)$ holds for all $x : X$. A crucial lemma in proving Pataraia's theorem says that every dcpo has a greatest inflationary endomap, which leads us to the following definition and theorem.

**Definition 4.** Let $\mathcal{V}$, $\mathcal{U}$ and $\mathcal{T}$ be universes. *Pataraia's-Lemma*$_{\mathcal{V},\mathcal{U},\mathcal{T}}$ asserts that every $\mathcal{V}$-dcpo $(D, \sqsubseteq)$ with $D : \mathcal{U}$ and $\sqsubseteq$ taking values in $\mathcal{T}$, has a greatest monotone inflationary endomap.

**Theorem 3.** Pataraia's-Lemma$_{\mathcal{V},\mathcal{V}^+ \sqcup \mathcal{U},\mathcal{V}}$ *implies* Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$.

Thus, the crucial lemma in the proof of Pataraia's fixed point theorem fails in predicative type theory. Whether Pataraia's fixed point theorem itself implies some resizing axiom is part of ongoing research on predicative domain theory in Univalent Foundations.

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