A refinement of Gabriel-Ulmer duality

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Overview

Context

- Meta-theory of type theory
- Homotopy-coherent algebraic structures

Motivation

Initiality $\Rightarrow$ "Type theory is algebraic" –
i.e. CwFs are algebraic –
i.e. they are models of a generalized/essentially algebraic theory –
i.e. they form a locally finitely presentable category.

What if we want to think about a **2-category** – or $(2, 1)$-category – of CwFs?

In this talk

- 'Clans' as a finer analysis of generalized algebraic theories by interpolating between finite-product theories and finite-limit theories
- Start by reviewing functorial semantics

Algebraic theories

Definition

A **single-sorted algebraic theory** (SSAT) is a pair (Σ, E) consisting of

- a family $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$, of sets of n-ary **operations**
- a set of **equations** E whose elements are pairs of open terms over Σ

Definition

The **syntactic category** $\mathcal{C}(\Sigma, E)$ of a SSAT is given as follows:

1. For each natural number $n \in \mathbb{N}$ there is an **object** $[n]$
2. **morphisms** $\sigma : [n] \to [m]$ are m-tuples of terms in n variables modulo E-provable equality
3. **identities** are lists of variables, **composition** is given by substitution

Theorem

Given a SSAT (Σ, E):

1. *$\mathcal{C}(\Sigma, E)$ has finite products given by $[n] \times [m] = [n + m]$*
2. $\mathbf{Mod}(\Sigma, E) \simeq \mathbf{FP}(\mathcal{C}(\Sigma, E), \mathbf{Set})$

Finite-product theories and finite-limit theories

Definition

- A **fp-theory** is just a small fp-category $\mathcal{C}$.
- **Models** of $\mathcal{C}$ are fp-functors $A : \mathcal{C} \to \mathbf{Set}$ (or into another fp-category).

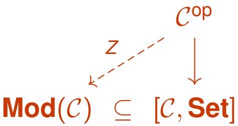
Denote the category of models by

$$\mathbf{Mod}(\mathcal{C}) \;:=\; \mathbf{FP}(\mathcal{C}, \mathbf{Set}) \overset{\text{full}}{\subseteq} [\mathcal{C}, \mathbf{Set}].$$

For every object $\Gamma \in \mathcal{C}$ of an fp-theory, the co-representable functor

$$\mathcal{C}(\Gamma, -) \;:\; \mathcal{C} \to \mathbf{Set}$$

is a model. Thus, the dual Yoneda embedding co-restricts to $\mathbf{Mod}(\mathcal{C})$.



Finite-limit theories

More expressive that finite-product theories are **finite-limit** theories:

Definition

- An **fl-theory** is a small finite-limit category $\mathcal{L}$.
- A **model** of $\mathcal{L}$ is a finite-limit preserving functor $A : \mathcal{L} \to \mathbf{Set}$.

Structures definable by finite-limit theories include

- categories, posets, 2-categories, monoidal categories, CwFs ...

Again $\mathcal{L}(\Gamma, -)$ is a model for every $\Gamma \in \mathcal{L}$ and we get an embedding

$$Z \ : \ \mathcal{L}^{\mathsf{op}} \ \to \ \mathbf{Mod}(\mathcal{L}) := \mathbf{FL}(\mathcal{L}, \mathbf{Set}) \overset{\text{full}}{\subseteq} [\mathcal{L}, \mathbf{Set}].$$

Moreover, we can characterize the essential image of Z in $\mathbf{Mod}(\mathcal{L})$.

Locally finitely presentable categories

Definition

- An object C of a cocomplete locally small category $\mathfrak{X}$ is called **compact**, if $\mathfrak{X}(C, -) : \mathfrak{X} \to \mathbf{Set}$ preserves filtered colimits.
- A category $\mathfrak{X}$ is called **locally finitely presentable**, if
  - $\mathfrak{X}$ is locally small and cocomplete
  - the full subcategory $\mathrm{comp}(\mathfrak{X}) \subseteq \mathfrak{X}$ on compact objects is essentially small and dense.

Theorem

- $\mathbf{Mod}(\mathcal{L})$ *is locally finitely presentable for all finite-limit theories* $\mathcal{L}$.
- *The essential image of* $Z : \mathcal{L}^{\mathrm{op}} \to \mathbf{Mod}(\mathcal{L})$ *comprises precisely the compact objects.*

Gabriel-Ulmer duality[1]

Theorem

There is a bi-equivalence of 2-categories

$$\mathbf{FL} \underset{\mathcal{L} \mapsto \mathbf{Mod}(\mathcal{L})}{\overset{\mathrm{comp}(\mathfrak{X})^{\mathrm{op}} \leftarrow\!\shortmid\, \mathfrak{X}}{\longleftarrow\!\!\!-\!\!\!-\!\!\!\longrightarrow}} \mathbf{LFP}^{\mathrm{op}}$$

where

- **FL** is the 2-category of **small** fl-categories and fl-functors
- **LFP** is the 2-category of locally finitely presentable categories and functors preserving all limits and filtered colimits.

[1] P. Gabriel and F. Ulmer. **Lokal präsentierbare Kategorien**. Vol. 221. Lecture Notes in Math. Springer-Verlag, 1971.

G-U duality for finite product theories

Definition

- A fl-category $\mathbb{R}$ is called **regular**, if
  - every morphism $f : A \to B$ factors into a regular epi followed by a mono
  - regular epis are stable under pullback
- $P \in \mathbb{R}$ is called **projective** if all regular epis $A \twoheadrightarrow P$ have sections
- $\mathbb{R}$ is **exact** if it is regular and all equivalence relations are kernel pairs

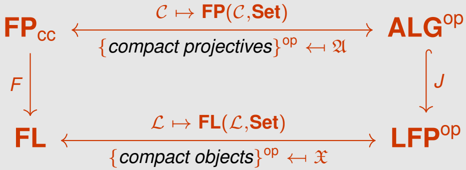
Definition

- A lfp category $\mathfrak{X}$ is called **algebraic**, if it is **exact**, and the compact projective objects are dense.
- An **algebraic functor** is functor $F : \mathfrak{X} \to \mathfrak{Y}$ between algebraic categories which preserves small limits, filtered colimits, and regular epis.

Duality for finite-product theories[2]

Theorem

*Gabriel-Ulmer duality 'restricts' to a duality between **small Cauchy-complete finite-product categories** and **algebraic categories**, as in the following diagram.*



- $\mathbf{FP}_{cc}$ *is the 2-category of Cauchy-complete finite-product categories*
- $\mathbf{ALG}$ *is the 2-category of algebraic categories and **algebraic functors***
- J *is the obvious (non-full) inclusion*
- F *is left biadjoint to the inclusion* $\mathbf{FL} \to \mathbf{FP}_{cc}$

[2] J. Adámek, J. Rosický, and E.M. Vitale. **Algebraic theories: a categorical introduction to general algebra**. Vol. 184. Cambridge University Press, 2010.

Toward clans

- Finite-limit theories have a nice duality theory but seem far from syntax
- Syntactic counterparts are given by
  - Freyd's **essentially algebraic theories**[3]
  - Cartmell's **generalized algebraic theories**[4]
- Clans can be viewed as a **less abstract** (i.e. closer to syntax) categorical representation of generalized algebraic theories

[3] P. Freyd. "Aspects of topoi". In: **Bulletin of the Australian Mathematical Society** 7.1 (1972), pp. 1–76.

[4] J. Cartmell. "Generalised algebraic theories and contextual categories". In: **Annals of pure and applied logic** 32 (1986), pp. 209–243.

Definition

A **clan** is a pair $(\mathcal{T}, \mathcal{T}_\dagger)$ consisting of

- a small category $\mathcal{T}$ with terminal object 1, and
- a set $\mathcal{T}_\dagger \subseteq \mathrm{mor}(\mathcal{T})$ of morphisms – called **display maps** – such that
  1. pullbacks of display maps along all maps exist & are display maps,

$$\begin{array}{ccc} \Delta^+ & \xrightarrow{s^+} & \Gamma^+ \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta & \xrightarrow{s} & \Gamma \end{array}$$

  2. display maps are closed under composition, and
  3. isomorphisms and terminal projections $\Gamma \twoheadrightarrow 1$ are display maps

- Definition (w/o smallness condition) due to Taylor[5] who called clans **categories with a class of display maps**.
- The name clan was suggested by Joyal[6]
- Clans can be viewed as a non-strict version of CwFs / CwAs / natural models which admit Σ-types and unit types and are 'democratic'.

[5] P. Taylor. "Recursive domains, indexed category theory and polymorphism". PhD thesis. University of Cambridge, 1987, § 4.3.2.

[6] A. Joyal. "Notes on clans and tribes". In: **arXiv preprint arXiv:1710.10238** (2017).

Examples

- Finite-product theories $\mathcal{C}$ can be viewed as clans by setting

$$\mathcal{C}_\dagger = \{\text{product projections}\}$$

- Finite-limit theories $\mathcal{L}$ can be viewed as clans by setting

$$\mathcal{L}_\dagger = \mathrm{mor}(\mathcal{L}).$$

We call such clans **fp-clans**, and **fl-clans**, respectively.

- Clan for categories

$$\mathcal{K} = \{\text{categories free on finite graphs}\}^{\mathsf{op}} \subseteq \mathbf{Cat}^{\mathsf{op}}$$
$$\mathcal{K}_\dagger = \{\text{functors induced by graph inclusions}\}^{\mathsf{op}}$$

$\mathcal{K}$ can be viewed as syntactic category of a generalized algebraic theory of categories with a sort O of objects, and a dependent sort $x, y{:}O \vdash M(x, y)$ of morphisms – vertices of a finite graph are object variables and edges are morphism variables in a context. Graph inclusions are dual to context extensions.

Models

Definition

A **model** of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \mathbf{Set}$ which preserves 1 and pullbacks of display-maps.

- The category $\mathbf{Mod}(\mathcal{T}) \subseteq [\mathcal{T}, \mathbf{Set}]$ of models is lfp.
- We have an inclusion functor $Z : \mathcal{T}^{\mathrm{op}} \to \mathbf{Mod}(\mathcal{T})$.
- For fp-clans $(\mathcal{C}, \mathcal{C}_\dagger)$ we have $\mathbf{Mod}(\mathcal{C}, \mathcal{C}_\dagger) = \mathbf{FP}(\mathcal{C}, \mathbf{Set})$.
- For fl-clans $(\mathcal{L}, \mathcal{L}_\dagger)$ we have $\mathbf{Mod}(\mathcal{L}, \mathcal{L}_\dagger) = \mathbf{FL}(\mathcal{L}, \mathbf{Set})$.
- $\mathbf{Mod}(\mathcal{K}, \mathcal{K}_\dagger) = \mathbf{Cat}$.

The weak factorization system

Would like a duality between clans and their categories of models.

Since the same lfp category can be represented by different clans, we cannot hope to reconstruct the clan from the models alone.

To this end, we equip the models with additional structure in form of a weak factorization system (also described by Simon Henry in recent HoTTest seminar).

The weak factorization system

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Definition

Let $\mathcal{T}$ be a clan. Define wfs $(\mathcal{E}, \mathcal{F})$ on $\mathbf{Mod}(\mathcal{T})$ by

- $\mathcal{F} := \mathbf{RLP}(\{Z(p) \mid p \in \mathcal{T}_\dagger\})$ class of **full maps**
- $\mathcal{E} := \mathbf{Cell}(\{Z(p) \mid p \in \mathcal{T}_\dagger\}) = \mathbf{LLP}(\mathcal{F})$ class of **extensions**

Call $A \in \mathbf{Mod}(\mathcal{T})$ a 0**-extension**, if $(0 \to A) \in \mathcal{E}$.

Full maps

- $f : A \to B$ in $\mathbf{Mod}(\mathcal{T})$ is full iff naturality squares

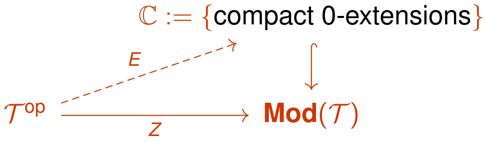
$$\begin{array}{ccc} A(\Delta) & \xrightarrow{f_\Delta} & B(\Delta) \\ {\scriptstyle A(p)}\downarrow & & \downarrow{\scriptstyle B(p)} \\ A(\Gamma) & \xrightarrow{f_\Gamma} & B(\Gamma) \end{array}$$

  are weak pullbacks for all display maps $p : \Delta \to \Gamma$.
- In particular, full maps are surjective and hence regular epis.
- For fl-clans, all maps are full.
- For fp-clans, the full maps are precisely the regular epis, whence

$$0\text{-extension} \;=\; \text{projective object}$$

Reconstructing the clan

Observation: $Z : \mathcal{T}^{\mathrm{op}} \to \mathbf{Mod}(\mathcal{T})$ sends display maps to extensions and objects to compact 0-extensions.



Theorem

$\mathbb{C}$ is a coclan (with extensions as display maps), and E exhibits $\mathbb{C}$ as Cauchy-completion of $\mathcal{T}^{\mathrm{op}}$.

Proof idea

Given $A \in \mathbb{C}$, fully cover it by a filtered colimit of objects in $\mathcal{T}$.

$$\operatorname{colim}(E \circ D) \xrightarrow{f} A \qquad D : \mathbb{I} \to \mathcal{T}^{\mathsf{op}}$$

Proof idea

Given $A \in \mathbb{C}$, fully cover it by a filtered colimit of objects in $\mathcal{T}$.

$$\begin{array}{c} \operatorname{colim}(E \circ D) \\ {\scriptstyle s}\uparrow\ \downarrow{\scriptstyle f} \\ A \end{array} \qquad\qquad D : \mathbb{I} \to \mathcal{T}^{\mathrm{op}}$$

- f splits since A is a 0-extension

Proof idea

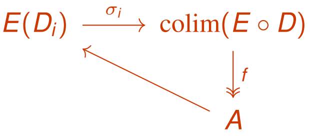
Given $A \in \mathbb{C}$, fully cover it by a filtered colimit of objects in $\mathcal{T}$.

$$\begin{array}{ccc} E(D_i) & \xrightarrow{\sigma_i} & \mathrm{colim}(E \circ D) \\ & \nwarrow & \big\downarrow f \\ & & A \end{array} \qquad D : \mathbb{I} \to \mathcal{T}^{\mathrm{op}}$$

- f splits since A is a 0-extension
- s factors through one of the colimit-injections, since A is compact

Proof idea

Given $A \in \mathbb{C}$, fully cover it by a filtered colimit of objects in $\mathcal{T}$.



$$D : \mathbb{I} \to \mathcal{T}^{\mathrm{op}}$$

- f splits since A is a 0-extension
- s factors through one of the colimit-injections, since A is compact
- D exists by the **fat small object argument**:

  - M. Makkai, J. Rosicky, and L. Vokrinek. “On a fat small object argument”. In: **Advances in Mathematics** 254 (2014), pp. 49–68

Clan-algebraic categories

Definition

A **clan-algebraic category** is an lfp category $\mathfrak{A}$ with an wfs $(\mathcal{E}, \mathcal{F})$ that arises as category of models of a clan.

Can we characterize clan-algebraic categories more abstractly?

Necessary conditions are that

- compact 0-extensions are dense in $\mathfrak{A}$, and
- $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps between compact 0-extensions,

but that's not enough:

Example

Consider

- $\mathbf{Inj} \subseteq [2^{\mathsf{op}}, \mathbf{Set}]$ full subcategory on injections
- $(\mathcal{E}, \mathcal{F})$ cofib. generated by $\{(0 \to Y0), (0 \to Y1)\}$

Then $\mathbf{Mod}(\{\text{compact 0-extensions}\}^{\mathsf{op}}) \simeq [2^{\mathsf{op}}, \mathbf{Set}]$.

Need additional 'exactness condition' depending on $(\mathcal{E}, \mathcal{F})$!

Characterization of clan-algebraic categories

Theorem

A lfp category $\mathfrak{A}$ with wfs $(\mathcal{E}, \mathcal{F})$ is clan-algebraic, if

1. $\mathbb{C} = \{$*compact* 0*-extensions*$\}$ *is dense in* $\mathfrak{A}$,
2. $(\mathcal{E}, \mathcal{F})$ *is cofibrantly generated by* $\mathcal{E} \cap \mathrm{mor}(\mathbb{C})$, *and*
3. *for* $C \in \mathbb{C}$ *and* $A \in \mathbf{Mod}(\mathbb{C}^{\mathsf{op}})$, *the functor* $\mathfrak{A}(C, -) : \mathfrak{A} \to \mathbf{Set}$ *preserves the colimit of the diagram* $\int A \to \mathbb{C} \to \mathfrak{A}$.

Quotients of componentwise-full equivalence relations

Lemma

Let $\mathcal{T}$ be a clan, and let $r = \langle r_1, r_2 \rangle : R \to A \times A$ an equivalence relation in **Mod**$(\mathcal{T})$ *with r_1 and r_2 full. Then r has a* ***full and effective*** *quotient, i.e. there is a full map $e : A \twoheadrightarrow B$ and a pullback square*

$$\begin{array}{ccc} R & \xrightarrow{r_1} & A \\ {\scriptstyle r_2}\downarrow & \lrcorner & \downarrow{\scriptstyle e} \\ A & \xrightarrow{e} & B \end{array} .$$

Conjecture

Condition 3 of the theorem is implied by $\mathfrak{A}$ having full and effective quotients of componentwise-full equivalence relations.

Models in higher types

Let $\mathcal{S}$ be the ∞-topos of spaces/types.

Let $\mathcal{C}_{\mathsf{Mon}}$ be the finite-product theory of monoids, and let $\mathcal{L}_{\mathsf{Mon}}$ be the finite-limit theory of monoids. Then

$$\mathbf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathbf{Set}) \simeq \mathbf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathbf{Set})$$

but $\mathbf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$ and $\mathbf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S})$ are different:

- $\mathbf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S})$ is just the category of monoids
- $\mathbf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$ is the category 'A_∞-algebras', i.e. homotopy-coherent monoids.

Moral

By being 'slimmer', finite-product theories leave room for higher coherences when interpreted in higher types.

This phenomenon has been discussed under the name 'animation' in:

- K. Cesnavicius and P. Scholze. "Purity for flat cohomology". In: **arXiv preprint arXiv:1912.10932** (2019)

Four clans for categories

Cat admits several clan-algebraic weak factorization systems:

- $(\mathcal{E}_1, \mathcal{F}_1)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2})\}$
- $(\mathcal{E}_2, \mathcal{F}_2)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (2 \to 1)\}$
- $(\mathcal{E}_3, \mathcal{F}_3)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2})\}$
- $(\mathcal{E}_4, \mathcal{F}_4)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2}), (2 \to 1)\}$

where $\mathbb{P} = (\bullet \rightrightarrows \bullet)$.

The right classes are:

$$\mathcal{F}_1 = \{\text{full and surjective-on-objects functors}\}$$
$$\mathcal{F}_2 = \{\text{full and bijective-on-objects functors}\}$$
$$\mathcal{F}_3 = \{\text{fully faithful and surjective-on-objects functors}\}$$
$$\mathcal{F}_4 = \{\text{isos}\}$$

Note that $\mathcal{F}_3$ is the class of trivial fibrations for the canonical model structure on **Cat**.

Four clans for categories

These correspond to the following clans:

$\mathcal{T}_1 = \{\text{free cats on fin. graphs}\}^{\text{op}}$ $\quad \mathcal{T}_1^{\dagger} = \{\text{graph inclusions}\}$

$\mathcal{T}_2 = \{\text{free cats on fin. graphs}\}^{\text{op}}$ $\quad \mathcal{T}_2^{\dagger} = \{\text{injective-on-edges maps}\}$

$\mathcal{T}_3 = \{\text{f.p. cats}\}^{\text{op}}$ $\quad \mathcal{T}_3^{\dagger} = \{\text{injective-on-objects functors}\}$

$\mathcal{T}_4 = \{\text{f.p. cats}\}^{\text{op}}$ $\quad \mathcal{T}_4^{\dagger} = \{\text{all functors}\}$

Models in higher types:

$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_1) = \{\text{Segal spaces}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_2) = \{\text{Segal categories}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_3) = \{\text{pre-categories}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_4) = \{\text{discrete 1-categories}\}$$

Thanks for your attention!