(Truncated) Simplicial Models of Type Theory

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Outline

1. Introduction
2. Reflexive graphs
3. 2-truncated simplicial spaces
4. Perspectives

Introduction I

In order to develop *synthetic higher category theory*, Riehl and Shulman introduced a *Type Theory with Shapes* (RSTT) in [RS17]: MLTT with types of simplices, allowing for defining *synthetic* $(\infty, 1)$*-categories* as *complete Segal/Rezk types*.

As a main feature, RSTT postulates *extension types*, i.e. for shape inclusions $\Phi \rightarrowtail \Psi$, families $A : \Phi \to \mathcal{U}$, and terms $a : \prod_{t:\Phi} A(t)$ there exists the type of liftings

$$\left\langle \prod_{t:\Psi} A(t) \Big|^{\Phi}_{a} \right\rangle \triangleq \left\{ \begin{array}{ccc} \Phi & \xrightarrow{a} & A \\ \rotatebox[origin=c]{-90}{$\rightarrowtail$} & \overset{\bar{a}}{\nearrow} & \\ \Psi & & \end{array} \right\}$$

Example & Definition: For a type A and terms $x, y : A$, define the *hom-types*

$$\hom_A(x, y) := \left\langle \prod_{t:\Delta^1} A(t) \Big|^{\partial \Delta^1}_{[x,y]} \right\rangle.$$

Introduction II

Goal: Understanding simplicial type theory and its models with the aim of doing synthetic $(\infty, 1)$-category theory with appropriate universes.

There is a model of RSTT in *simplicial spaces*, i.e. the model category $[\mathbb{\Delta}^{\mathrm{op}}, \mathbf{sSet}_{\mathrm{Quillen}}]_{\mathrm{Reedy}}$. This model structure presents the $(\infty, 1)$-category $\mathrm{PSh}_\infty(\mathbb{\Delta})$.

We consider variations such as $\mathrm{PSh}_\infty(\mathbb{\Delta}_{\leq k})_{\mathrm{Reedy}}$ for $k = 1, 2$.

We get a model of RSTT whenever we have Δ^1 together with $\leq\colon \Delta^1 \times \Delta^1 \to \mathrm{Prop}$ as an interval type.

Intended model: simplicial spaces I

Definitions from [RS17]:

- A type A is a *Segal type* if $(\Delta^2 \to A) \xrightarrow{\simeq} (\Lambda_1^2 \to A)$.
- A Segal type A is a *Rezk type* if $\mathrm{idtoiso}_A : \prod_{x,y:A} \mathrm{Id}_A(x,y) \xrightarrow{\simeq} \mathrm{iso}_A(x,y)$.
- A type A is a *discrete type* if $\mathrm{idtorarr}_A : \prod_{x,y:A} \mathrm{Id}_A(x,y) \xrightarrow{\simeq} \mathrm{hom}_A(x,y)$.

RSTT with a univalent universe can be modeled on $[\mathbb{\Delta}^{\mathrm{op}}, \mathbf{sSet}_{\mathrm{Quillen}}]_{\mathrm{Reedy}}$, cf. [Shu15].

The notions just introduced semantically coincide with their classical analogues, *at the level of objects*.

Intended model: simplicial spaces II

- Types are interpreted as Reedy fibrant objects. Families of types are interpreted as Reedy fibrations. A map f is a fibration if $m \perp f$ for all m which are componentwise trivial cofibrations in $\mathbf{sSet}$.
- Segal types are interpreted as Segal spaces, i.e. Reedy fibrant objects X with $m \otimes I(i) \perp X$ for all monomorphisms m, and $i : \Lambda^2_1 \rightarrowtail \Delta^2$, $I : \mathbf{sSet} \hookrightarrow [\mathbb{\Delta}^{\mathrm{op}}, \mathbf{sSet}]$. *Segal types are ∞-precategories (i.e. non-univalent).*
- Rezk types are interpreted as complete Segal spaces, *aka* Rezk spaces, i.e. Segal spaces X where $X_0 \simeq X_{\mathrm{hoeq}}$. *Rezk types are univalent ∞-categories.*
- Discrete types are Rezk types X such that all X_n are discrete simplicial sets. *Discrete types are (univalent) ∞-groupoids.*

Subuniverses of simplicial spaces I

In RSTT, define

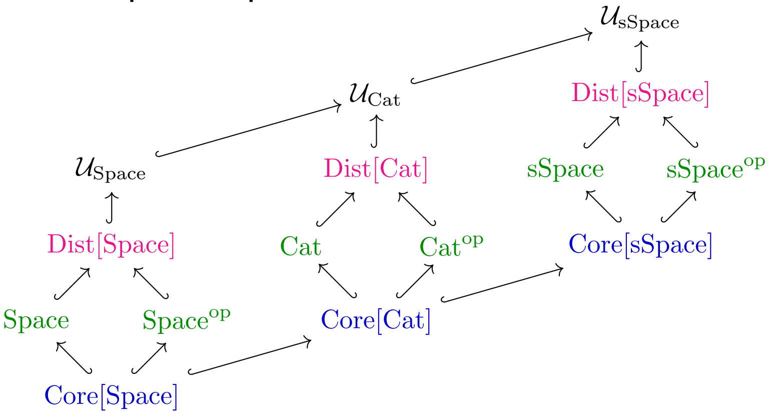
$$\begin{aligned}\mathrm{isSegal}(A) &:\equiv \mathrm{isEquiv}\big((\Delta^2 \to A) \to (\Lambda_1^2 \to A)\big),\\ \mathrm{isRezk}(A) &:\equiv \mathrm{isSegal}(A) \times \mathrm{isEquiv}(\mathrm{idtoiso}_A),\\ \mathrm{isDisc}(A) &:\equiv \mathrm{isEquiv}(\mathrm{idtorarr}_A) \simeq \mathrm{isRezk}(A) \times \prod_{x,y:A} \prod_{f:\hom_A(x,y)} \mathrm{isIso}(f),\end{aligned}$$

giving rise to contexts

$$\begin{aligned}\mathrm{Segal} &:= [\![A : \mathcal{U}, p : \mathrm{isSegal}(A)]\!],\ \mathrm{Rezk} := [\![A : \mathcal{U}, p : \mathrm{isRezk}(A)]\!],\\ \mathrm{Disc} &:= [\![A : \mathcal{U}, p : \mathrm{isDisc}(A)]\!],\end{aligned}$$

in either the “full” simplicial space model $\mathrm{PSh}_\infty(\mathbb{\Delta})$, or the truncated ones $\mathrm{PSh}_\infty(\mathbb{\Delta}_{\leq k})$.

Subuniverses of simplicial spaces II



sSpace := CSS of simpl. spaces

$\text{Cat}_n := (\mathcal{U}_{\text{Cat}})^*(\text{sSpace}_n) \simeq \Gamma(\text{coCart}(\Delta^n))$

$\text{Space}_n := \Gamma(\text{LFib}(\Delta^n)) \simeq (\mathcal{U}_{\text{Space}})^*(\text{Cat}_n)$

Where are Disc and Rezk?

$\text{Dist}[\text{sSpace}]_n := \Gamma(\text{ExpFib}(\Delta^n))$

$\text{Dist}[\text{Cat}]_n := (\mathcal{U}_{\text{Cat}})^*(\text{Dist}[\text{sSpace}]_n)$

$\text{Dist}[\text{Space}]_n := (\mathcal{U}_{\text{Space}})^*(\text{Dist}[\text{Cat}]_n)$

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Reflexive graphs I

Taking $\mathrm{PSh}_\infty(\mathbb{\Delta}_{\leq 1})$ gives rise to the reflexive graph model, already investigated by Rijke-Spitters [Rij17]:

Contexts Γ

$$\Gamma_0 : \mathcal{U}$$
$$\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$$
$$\rho_\Gamma : \prod_{x:\Gamma_0} \Gamma_1(x,x)$$

Context substitution $\varphi : \Gamma \to \Delta$

$$\varphi_0 : \Gamma_0 \to \Delta_0$$
$$\varphi_1 : \prod_{\{x,y:\Gamma_0\}} \Gamma_1(x,y) \to \Delta_1(\varphi_0 x, \varphi_0 y)$$
$$\rho_\varphi : \prod_{x:\Gamma_0} (\varphi_1(\rho_\Gamma x) = \rho_\Delta(\varphi_0 x))$$

Families $\Gamma \vdash A$

$$A_0 : \Gamma_0 \to \mathcal{U}$$
$$A_1 : \prod_{\{x,y:\Gamma_0\}} \Gamma_1(x,y) \to A_0(x) \to A_0(y) \to \mathcal{U}$$
$$\rho_A : \prod_{\{x:\Gamma_0\}} \prod_{u:A_0(x)} A_1(\rho_\Gamma(x), u, u)$$

Terms $a : A$

$$a_0 : \prod_{x:\Gamma_0} A_0(x)$$
$$a_1 : \prod_{\{x,y:\Gamma_0\}} \prod_{e:\Gamma_1(x,y)} A_1(e, a_0(x), a_0(y))$$
$$\rho_a : \prod_{x:\Gamma_0} \big(a_1(\rho_\Gamma x) = \rho_A(a_0 x)\big)$$

We obtain a model of (cohesive) *simplicial type theory* on top of $\mathsf{MLTT}_{\Pi,\Sigma,\mathrm{Id},\mathcal{U}}$, with Δ^n as types.

Disc as a reflexive graph

Theorem

$[\![\mathrm{Disc}]\!]_1$ *is equivalent to the reflexive graph* Disc_1 *with:*

$$
\begin{aligned}
(\mathrm{Disc}_1)_0 &:\equiv \mathrm{Set} \\
(\mathrm{Disc}_1)_1\ X\ Y &:\equiv X \to Y \to \mathrm{Prop} \\
\rho_{\mathrm{Disc}_1}\ X &:\equiv \lambda xy.\,(x = y)
\end{aligned}
$$

Proof idea: For $A : \mathcal{U}$ we have $\mathrm{isDisc}(A) \simeq \prod_{a:A} \mathrm{isContr}(\mathrm{arrfrom}_A(a))$. This leads to

$$
\begin{aligned}
[\![\mathrm{Disc}]\!]_1\, X\, Y &\simeq \sum_{Z:X\to Y\to\mathcal{U}}\ \prod_{\{x:X,\ y:Y\}}\ \prod_{z:Z\,x\,y} \mathrm{isContr}(Z\,x\,y) \\
&\simeq \sum_{Z:X\to Y\to\mathcal{U}}\ \prod_{\{x:X,\ y:Y\}} \mathrm{isProp}(Z\,x\,y).
\end{aligned}
$$

Rezk as a reflexive graph

Theorem

$[\![\mathrm{Rezk}]\!]_1$ *is equivalent to the reflexive graph* Rezk_1 *with:*

$$\begin{aligned} (\mathrm{Rezk}_1)_0 &:\equiv \mathrm{Poset} \\ (\mathrm{Rezk}_1)_1\ X\ Y &:\equiv \text{“\textit{full relations} } X \to Y \to \mathrm{Prop}\text{”} \\ \rho_{\mathrm{Rezk}_1}\ X &:\equiv X.\,\mathrm{PO} \end{aligned}$$

Hence, Disc_1 is *not* Rezk.

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2-truncated simplicial spaces I

Contexts Γ

$$\Gamma_0 : \mathcal{U}$$
$$\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$$
$$\rho_\Gamma : \prod_{x:\Gamma_0} \Gamma_1(x,x)$$

$$\Gamma_{\mathrm{comp}} : \prod_{x,y,z:\Gamma_0} \Gamma_1(x,y) \to \Gamma_1(y,z) \to \Gamma_1(x,z) \to \mathcal{U}$$
$$\Gamma_{\mathrm{left}} : \prod_{x,y:\Gamma_0} \prod_{f:\Gamma_1(x,y)} \Gamma_{\mathrm{comp}}(\rho_\Gamma x, f, f)$$
$$\Gamma_{\mathrm{right}} : \prod_{x,y:\Gamma_0} \prod_{f:\Gamma_1(x,y)} \Gamma_{\mathrm{comp}}(f, \rho_\Gamma y, f)$$
$$\Gamma_{\mathrm{coh}} : \prod_{x:\Gamma_0} \big(\Gamma_{\mathrm{left}}(\rho_\Gamma x) = \Gamma_{\mathrm{right}}(\rho_\Gamma x)\big)$$

Context substitution $\varphi : \Gamma \to \Delta$

$$\varphi_0 : \Gamma_0 \to \Delta_0$$
$$\varphi_1 : \prod_{\{x,y:\Gamma_0\}} \Gamma_1(x,y) \to \Delta_1(\varphi_0 x, \varphi_0 y)$$
$$\rho_\varphi : \prod_{x:\Gamma_0} (\varphi_1(\rho_\Gamma x) = \rho_\Delta(\varphi_0 x))$$

$$\varphi_{\mathrm{comp}} : \prod_{\{\cdots\}} \Gamma_{\mathrm{comp}}(f,g,h) \to \Delta_{\mathrm{comp}}(\varphi_1 f, \varphi_1 g, \varphi_1 h)$$
$$\varphi_{\mathrm{left}} : \prod_{\{x,y:\Gamma_0\}} \prod_{f:\Gamma_1(x,y)} (\varphi_{\mathrm{comp}}(\Gamma_{\mathrm{left}}(f)) \underset{\rho_\varphi(x)}{=} \Delta_{\mathrm{left}}(\varphi_1 f))$$
$$\varphi_{\mathrm{right}} : \prod_{\{x,y:\Gamma_0\}} \prod_{f:\Gamma_1(x,y)} (\varphi_{\mathrm{comp}}(\Gamma_{\mathrm{right}}(f)) \underset{\rho_\varphi(y)}{=} \Delta_{\mathrm{right}}(\varphi_1 f))$$
$$\varphi_{\mathrm{coh}} : \prod_{x:\Gamma_0} (\varphi_{\mathrm{left}}(\rho_\Gamma x) \underset{\cdots}{=} \varphi_{\mathrm{right}}(\rho_\Gamma x))$$

2-truncated simplicial spaces II

Families $\Gamma \vdash A$

$$A_0 : \Gamma_0 \to \mathcal{U}$$

$$A_1 : \prod_{\{x,y:\Gamma_0\}} \Gamma_1(x,y) \to A_0(x) \to A_0(y) \to \mathcal{U}$$

$$\rho_A : \prod_{\{x:\Gamma_0\}} \prod_{u:A_0(x)} A_1(\rho_\Gamma(x), u, u)$$

$$A_{\text{comp}} : \prod_{\{\cdots\}} \Gamma_{\text{comp}}(f,g,h) \to A_1(a,b) \to A_1(b,c) \to A_1(a,c) \to \mathcal{U}$$

$$A_{\text{left}} : \prod_{\{\cdots\}} \cdots \to A_{\text{comp}}(\Gamma_{\text{left}} f, \rho_A x, g, g)$$

$$A_{\text{right}} : \prod_{\{\cdots\}} \cdots \to A_{\text{comp}}(\Gamma_{\text{right}} f, g, \rho_A y, g)$$

$$A_{\text{coh}} : \cdots$$

Terms $a : A$

$$a_0 : \prod_{x:\Gamma_0} A_0(x)$$

$$a_1 : \prod_{\{x,y:\Gamma_0\}} \prod_{e:\Gamma_1(x,y)} A_1(e, a_0(x), a_0(y))$$

$$\rho_a : \prod_{x:\Gamma_0} \big(a_1(\rho_\Gamma x) = \rho_A(a_0 x)\big)$$

$$a_{\text{comp}} : \prod_{\cdots, \sigma:\Gamma_{\text{comp}}(f,g,h), \cdots} A_{\text{comp}}(\sigma, a_1(f), a_1(g), a_1(h))$$

$$a_{\text{left}} : \prod_{\{\cdots\}} \cdots \to A_{\text{left}}(g) =_{\rho_a x} a_{\text{comp}}(\Gamma_{\text{left}} f)$$

$$a_{\text{right}} : \prod_{\{\cdots\}} \cdots \to A_{\text{right}}(g) =_{\rho_a y} a_{\text{comp}}(\Gamma_{\text{right}} f)$$

$$a_{\text{coh}} : \cdots$$

Disc and Rezk as 2-truncated simplicial spaces

$[\![\mathrm{Disc}]\!]_2$ has as vertices 1-types, as edges Set-valued relations, and as triangles 2-spans of propositions.

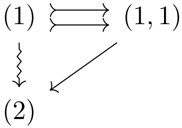
We conjecture $[\![\mathrm{Rezk}]\!]_2$ to have as vertices 1-categories. The edges are some form of "generalized distributors".

Extrapolating to dimension n, one might expect $[\![\mathrm{Disc}]\!]_n$ to have as vertices the $(n-1)$-types, as edges $(n-2)$-type valued spans, as triangles $(n-3)$-type valued 2-spans etc.

We suspect $[\![\mathrm{Rezk}]\!]_n$ to have as vertices $(n-1,1)$-categories, as edges "generalized $(n-2)$-type valued distributors", as triangles "$(n-3)$-type valued 2-distributors (?)" etc.

Direct replacement

Direct replacement of $\mathbb{\Delta}_{\leq 1}$, construction due to Szumiło, cf. [Szu14], [KS17]:



A context Γ is given by the following data:

$$\Gamma_0 : \mathcal{U}$$

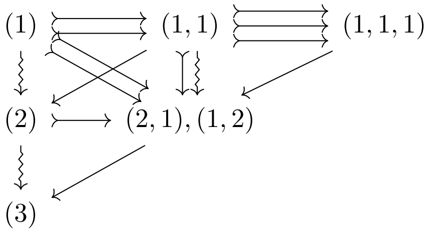
$$\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$$

$$\Gamma_{\text{refl}} : \prod_{x:\Gamma_0} \Gamma_1(x, x) \to \mathcal{U}$$

$$\sigma_\Gamma : \text{isEquiv}\Big(\Big(\sum_{x:\Gamma_0} \sum_{f:\Gamma_1(x,x)} \Gamma_{\text{refl}}(x, f)\Big) \to \Gamma_0\Big)$$

$$\simeq \prod_{x:\Gamma_0} \text{isContr}\Big(\sum_{f:\Gamma_1(x,x)} \Gamma_{\text{refl}}(x, f)\Big)$$

Direct replacement of $\mathbb{\Delta}_{\leq 2}$:



$$\Gamma_0 : \mathcal{U}$$

$$\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$$
$$\Gamma_2 : \prod_{x,y,z:\Gamma_0} \Gamma_1(x,y) \to \Gamma_1(y,z) \to \Gamma_1(x,z) \to \mathcal{U}$$
$$\Gamma_{\text{refl}} : \prod_{x:\Gamma_0} \Gamma_1(x,x) \to \mathcal{U}$$
$$\Gamma_{\text{left}} : \prod_{\{x,y:\Gamma_0\}} \prod_{f:\Gamma_1(x,y)} \prod_{g:\Gamma_1(y,y)} \Gamma_{\text{comp}}(f,g,f) \to \mathcal{U}$$
$$\Gamma_{\text{right}} : \prod_{\{x,y:\Gamma_0\}} \prod_{f:\Gamma_1(x,x)} \prod_{g:\Gamma_1(x,y)} \Gamma_{\text{comp}}(f,g,g) \to \mathcal{U}$$
$$\Gamma_{\text{triangle}} : \ldots$$

$$\sigma_\Gamma : \prod_{x:\Gamma_0} \text{isContr}\Big(\sum_{f:\Gamma_1(x,x)} \Gamma_2(x,f) \Big)$$

+ witnesses for degeneration

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Perspectives

Given a predicate $P : X \to \mathrm{Prop}$ internally in RSTT, is there a way to define the corresponding full predicate? (Would yield definitions of $\mathcal{U}_{\mathrm{Space}}, \mathcal{U}_{\mathrm{Cat}}$.)

Can we define Cat as the intersection of Rezk and sSpace?

If we define $\mathrm{sSpace} := (\mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Space})$, what would be the element fibration?

Externally, universes and notions of fibrations are interdefinable. What about internally?

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