Cubical Assemblies and the Independence of the Propositional Resizing Axiom

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Overview

I present a model of cubical type theory [Cohen et al., 2018] in the category of internal cubical objects in the category of assemblies. This model

- has a univalent and *impredicative* universe;
- does NOT satisfy the propositional resizing axiom.

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Cubical Type Theory

Cubical type theory [Cohen et al., 2018] is a type theory intended for constructive justifications of the univalence axiom and higher inductive types. The univalence axiom is provable in cubical type theory.

Impredicative Universes

We say a universe $\mathcal{U}$ is *impredicative* if, for any type A and type family $B : A \to \mathcal{U}$, the dependent product type $\prod_{x:A} B(x)$ belongs to $\mathcal{U}$.

Examples

- $(\prod_{X:\mathcal{U}} X \to X) : \mathcal{U}$
- $(\prod_{X:\mathcal{U}} X \to (X \to X) \to X) : \mathcal{U}$
- $(\prod_{X:\mathcal{U}} \prod_{x:X} x = x \to X) : \mathcal{U}$

Impredicative Encodings of (Higher) Inductive Types

An interesting use of such an impredicative universe is to represent data types as polymorphic function types [Shulman, 2011].

- $\mathbb{N} :\equiv \prod_{X:\mathcal{U}} X \to (X \to X) \to X$
- $\mathbb{S}^1 :\equiv \prod_{X:\mathcal{U}} \prod_{x:X} x = x \to X$
- $\|A\| :\equiv \prod_{X:\mathsf{hProp}} (A \to X) \to X$ where hProp is the universe of homotopy propositions in $\mathcal{U}$.

They have constructors and recursors in the sense of the HoTT Book, but do not satisfy the induction principle in general. Refinements of impredicative encodings are studied by Awodey, Frey and Speight [Awodey et al., 2018].

Propositional Resizing

We say a homotopy proposition A admits *propositional resizing* if there exist a homotopy proposition A' in $\mathcal{U}$ and an equivalence $A \simeq A'$.

- Originally proposed by Voevodsky [Voevodsky, 2012].
- A form of impredicativity for homotopy propositions.

If $\mathcal{U}$ is impredicative, we have a candidate for propositional resizing. For a homotopy proposition A, define $A^* :\equiv \prod_{X:\mathsf{hProp}}(A \to X) \to X$ and $\eta_A :\equiv \lambda a.\lambda Xh.ha : A \to A^*$.

Proposition

A homotopy proposition A admits propositional resizing if and only if the function $\eta_A : A \to A^*$ is an equivalence.

Outline

Overview

Internalizing cubical set models [Bezem et al., 2014, Cohen et al., 2018] in the category $\mathbf{Asm}$ of *assemblies*.

- $\mathbf{Asm}$ is a model of extensional dependent type theory with propositional truncation.
- $\mathbf{Asm}$ has an impredicative universe $\mathbf{PER}$.
- We define an internal category $\square$ of cubes in $\mathbf{Asm}$.
- Let $\mathbf{CAsm}$ be the category of internal presheaves on $\square$ which we call *cubical assemblies*.

Theorem

$\mathbf{CAsm}$ is a model of cubical type theory with an impredicative universe.

The Orton-Pitts Axioms

$\mathbf{CAsm}$ can be seen as an instance of more general construction of models of cubical type theory given by Orton and Pitts [Orton and Pitts, 2016]. They construct a model of cubical type theory in an elementary topos with

- an interval object $\mathbb{I}$; and
- an object $\mathsf{Cof} \rightarrowtail \Omega$ of *cofibrant propositions*

satisfying several axioms. In that model,

- a context Γ is an object of the topos;
- a type $\Gamma \vdash A$ is a morphism $A \to \Gamma$ equipped with a *composition structure*; and
- a term $\Gamma \vdash a : A$ is a section of the morphism $A \to \Gamma$.

The axioms can be written in extensional dependent type theory with propositional truncation but without the subobject classifier. So we can use their construction for making $\mathbf{CAsm}$ a model of cubical type theory, although $\mathbf{CAsm}$ is not an elementary topos.

Universes of Fibrant Types

The Orton-Pitts construction does not guarantee the existence of universes of fibrant types. We can use recent work by Licata, Orton, Pitts and Spitters [Licata et al., 2018] on constructing universes of fibrant types using the right adjoint to the exponential functor $(-)^{\mathbb{I}}$.

Outline

Propositional Resizing

Recall:

Proposition

In cubical type theory with an impredicative universe, a homotopy proposition $\Gamma \vdash A$ admits propositional resizing if and only if the function $\Gamma \vdash \eta_A : A \to A^*$ is an equivalence, where $A^* :\equiv \prod_{X:\mathsf{hProp}}(A \to X) \to X$.

Theorem

In the cubical assembly model, there exists a homotopy proposition $\Gamma \vdash A$ such that A^* is inhabited but A is not.

Overview

1. Find a family $\Gamma \vdash A$ of assemblies that is uniform and well-supported but not inhabited.
2. The codiscrete presheaf $\Delta\Gamma \vdash \nabla A$ over the constant presheaf $\Delta\Gamma$ is uniform and well-supported but not inhabited.
3. ∇A is a homotopy proposition.
4. If a family $\Delta \vdash \mathrm{B}$ in $\mathbf{CAsm}$ is uniform and well-supported, then B^* is inhabited.

Assemblies

- An *assembly* is a set A equipped with a non-empty set $E_A(a)$ of natural numbers for each $a \in A$. Elements of $E_A(a)$ are called *realizers* of a.
- A morphism $f : A \to B$ of assemblies is a function $f : A \to B$ between the underlying sets such that there exists a partial recursive function e such that, for any $a \in A$ and $n \in E_A(a)$, the application en is defined and belongs to $E_B(f(a))$.
- The category $\mathbf{Asm}$ of assemblies has an impredicative universe $\mathbf{PER}$ of partial equivalence relations on $\mathbb{N}$.
- A morphism $f : A \to \mathbf{PER}$ corresponds to a *modest family* $B \to A$: for any $a \in A$ and $b, b' \in B_a$, if $E(b) \cap E(b') \neq \emptyset$ then $b = b'$.

Uniform Objects

- An assembly A is *uniform* if $\bigcap_{a \in A} E(a)$ is non-empty.
- In a regular category, an object A is *well-supported* if the unique morphism $A \to 1$ is regular epi.

Proposition

For assemblies A and X, if A is uniform and well-supported and X is modest, then the morphism

$$\lambda x a.x : X \to (A \to X)$$

is an isomorphism.

Uniform Presheaves

Uniform and modest internal presheaves are defined pointwise.

Proposition

For internal presheaves A and X, if A is uniform and well-supported and X is modest, then the morphism

$$\lambda x a . x : X \to (A \to X)$$

is an isomorphism.

Corollary

If a type $\Gamma \vdash A$ in $\mathbf{CAsm}$ is uniform and well-supported, then $A^* \equiv \prod_{X:\mathsf{hProp}}(A \to X) \to X$ is inhabited.

Codiscrete Cubical Assemblies

For an assembly Γ, let $\Delta\Gamma$ denote the constant presheaf. For a family $\Gamma \vdash A$ of assemblies, we can define the *codiscrete cubical assembly* $\Delta\Gamma \vdash \nabla A$ in such a way that

- the points of ∇A is elements of A; and
- any two points are connected by a unique path.

Proposition

- If A is uniform then so is ∇A.
- If A is well-supported then so is ∇A.
- If ∇A is inhabited then so is A.

So our goal is to find a family $\Gamma \vdash A$ of assemblies that is uniform and well-supported but not inhabited.

The Counterexample

Define a family $\Gamma \vdash A$ of assemblies as follows.

- $\Gamma = (\mathbb{N}, n \mapsto \{m \in \mathbb{N} \mid m > n\})$
- $A(n) = (\{m \in \mathbb{N} \mid m > n\}, m \mapsto \{n, m\})$

Then A is uniform and well-supported but not inhabited, and thus $\Delta\Gamma \vdash \nabla A$ is a homotopy proposition in $\mathbf{CAsm}$ that does not admit propositional resizing.

Conclusion

We have got a model of cubical type theory in the category of cubical assemblies that

- has a univalent and impredicative universe; but
- does not satisfy the propositional resizing axiom.

Related and Future Work I

- Is there a model of type theory with a univalent and impredicative universe that satisfies the propositional resizing axiom?
  - Benno van den Berg [Berg, 2018] has constructed a model of type theory with a univalent and impredicative universe of 0-types that satisfies the propositional resizing axiom, but in his model computational rules for identity types and dependent product types hold only up to propositional equality.
- Higher inductive types via W-types with reductions [Swan, 2018b], including propositional truncation.
- Can we say that $\mathbf{CAsm}$ is a "realizability higher category"?
  - In $\mathbf{Asm}$ every proposition (monomorphism) is modest.
  - Church's Thesis: $\forall_{\mathsf{f}:\mathbb{N}\to\mathbb{N}}\exists_{e:\mathbb{N}}\varphi_e = \mathsf{f}$, where φ_e is the e-th partial recursive function. It does not hold in $\mathbf{CAsm}$ because the type of points is $\prod_{\mathsf{f}:\mathbb{N}\to\mathbb{N}}\sum_{e:\mathbb{N}}\varphi_e = \mathsf{f}$ in $\mathbf{Asm}$. Possibly some full subcategory of $\mathbf{CAsm}$ satisfies Church's Thesis (joint work with Andrew Swan).

Related and Future Work II

- Model structures
  - Frumin and van den Berg [Frumin and Berg, 2018] has constructed a model structure on a full subcategory of an elementary topos that satisfies similar conditions to those of Orton and Pitts. Their construction will work for cubical assemblies.
  - Trivial cofibration-fibration factorization on $\mathbf{CAsm}$ using Swan's small object argument in Grothendieck fibrations [Swan, 2018a, Swan, 2018b].
- How to relate Stekelenburg's constructive simplicial homotopy [Stekelenburg, 2016]?

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