Univalent Foundations and the Constructive View of Theories

Workshop on Homotopy Type Theory/ Univalent Foundations, Oxford

July 7, 2018

All significant foundational projects in mathematics of the past — including the nowadays standard set-theoretic foundations —/ have been strongly motivated and supported by reasoning outside the pure mathematics, which can be loosely called philosophical.

UF is not an exception. Vladimir's thinking behind his work in the foundations of maths also has strong pragmatic aspects.

Wuhan and Bangalore talks, Nov-Dec 2003

(available at Vladimir's IAS personal page; I quote:)

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- Computerized library of math knowledge — computerized version of Bourbaki;
- Connecting pure and applied mathematics.

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- The proposal of using UF as a representational formal framework outside the pure mathematics is mine, not Vladimir's.

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UF in its existing form satisfy all (?) these desiderata at certain extent (to be better evaluated and further improved).

Philosophical Thinking behind MLTT

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MLTT implements mathematically the idea of General Proof theory (Prawitz):
Proof = evidence, not just a syntactic derivation from axioms; proof-theoretic semantics vs. model-theoretic semantics.

Philosophical Thinking behind MLTT

"[P]roof and knowledge are the same. Thus, if proof theory is construed not in Hilbert's sense, as metamathematics, but simply as a study of proofs in the original sense of the word, then proof theory is the same as theory of knowledge, which, in turn, is the same as logic in the original sense of the word, as the study of reasoning, or proof, not as metamathematics." (Martin-Löf 1984)

The proof-checking feature of UF is an implementation of these ideas reinforced with the homotopical intuition.

WARNING: there is a point where the intended semantics of MLTT and HoTT diverge: to be discussed later on.

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UF-inspired architectures for KR may help us to solve this problem.

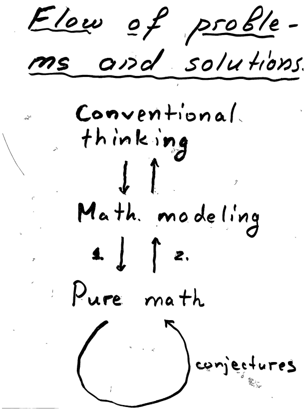
The Bridging Problem (Wuhan 2003)

"We discovered very fundamental classes of objects (eg. categories, sheaves, cohomology, simplicial sets). May be as fundamental as groups... but we do not use them to solve problems outside math."

Subdue Maths to Practical Needs is Not a Solution

"In order to apply mathematics to a practical problem effectively one should not in ones mathematical research try to focus on prospective applications in the real life but should do the opposite: to abstract yourself from the real life and look at the problem as at a formal game/puzzle."

Connection between Pure and Applied Maths according to VV (Bangalore 2003)



New flow chart:

Conventional thinking

↓ ↑

Computer modeling

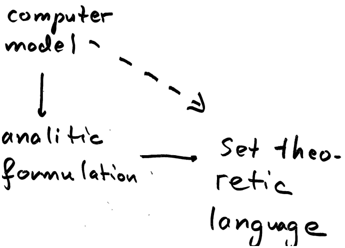
↓ ↑

|| Math. modeling

↓ ↑

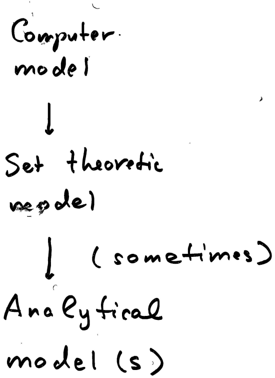
Pure math

Double Translation



(no UF yet at this point)

Shortcut from Computations to Foundations



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Shortcut from Computations to Foundations

Conclusion: the layer between computer modeling and pure math needs reorganization

- changes in education to underscore sequence

computer model → set theoretic model
↓
Analitic or other reformulation

(no UF yet at this point)

from Computations to Foundations to Computations

$$\mathit{Foundations} \underset{?}{\overset{UF}{\rightleftarrows}} \mathit{Computations} \underset{modeling}{\overset{engineering}{\rightleftarrows}} \mathit{Real\ Life}$$

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A different approach that I'm taking is to use UF as a formal representational framework for a wide range of knowledge including scientific theories.

What kind of theory can be built with the UF formal architecture?

I shall look into a relevant philosophical discussion, not into KR/CS.

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- Semantic (aka Non-Statement) View (since late 1950ies: P. Suppes, B. van Fraassen et al.): Tarskian formal semantics of Hilbert-style theories: a theory is identified with a class of models rather with any particular (interpreted) syntactic presentation.

Constructive Architecture

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- Gentzen-style rule-based architecture instead of the familiar Hilbert-style axiom-based architecture
- Makes formal rules theory- and subject-specific and informative in this sense. Such rules may not always qualify as *logical* under one's favourite conception of logicality. This is a very little explored dimension of the "axiomatic freedom" (that Hilbert himself to the best of my knowledge didn't consider seriously).

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- combines representations of knowledge-that and of knowledge-how into a single formal framework;
- supports thought-experimentation and the experimental design.

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- Euclid: Axioms (Common Notions) and (at least some) Postulates in Euclid are *rules* but not sentences that admit truth-values, i.e., not axioms in the modern sense. Many of Euclid's "Propositions" are Problems followed by Constructions while some other are Theorems followed by Proofs.

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- Newton's *Principia* Mathematical and experimental *methods* play a crucial role in the theoretical structure of the *Principia*. The title of the first Section of the first Book of Newton's *Principia* reads: *Of the Method of First and Last Ratios of Quantities*

Motivating Examples (continued)

- Quantum Field Theory: comprises both mathematical methods (such as Renormalization methods) and very sophisticated experimental methods used, in particular in ATLAS and CMS experiments at CERN's LHC in 2012.

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Do the experimental methods play a role in the *logical* structure of QFT? Yes, because they provide crucial *evidences* (proofs) for claims of this theory.

Why HoTT?

HoTT provides a novel unintended semantics for MLTT that distinguishes between propositional and non-propositional (higher) types. This feature

- supports the representation of extra-logical methods and operations in theories such as methods of conducting physical experiments;
- at the same time it makes explicit the logical relevance of such extra-logical operations as verifiers of corresponding sentences.

alternative explanations of $t : T$ in Martin-Löf 1984

- t is an element of set T (Curry-Horward)
- t is a proof (construction) of proposition T
- t is a method of fulfilling (realizing) the intention (expectation) T (Heyting)
- t is a method of solving the problem (doing the task) T (Kolmogorov)

propositions = sets

"If we take seriously , the idea that a proposition is defined by laying down how its canonical proofs are and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to unnecessary duplication to keep the notions of proposition and set [...] apart. Instead, we simply identify them." (Martin-Löf 1984)

Divergence between HoTT and the intended semantics of MLTT

The commulative *h*-hierarchy of types in HoTT restricts the interpretations to types as propositions and sets to appropriate *h*-levels:

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If A is a higher type I cannot see a justification for calling expression $a : A$ a *judgement*.
A suggested term borrowed from programming: a *declaration*.

Constructive View of Theories

A theory is a bunch of *methods* and a class (category) of their applications. Applications of methods are procedures, which bring about evidences supporting certain statements (via the propositional truncation). Application of a method is a matter of empirical test.

Conclusion:

The new dimension of the axiomatic freedom is worth to be explored. It promises to provide a lot of useful applications in KR.

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Is there a sense in which Hilbert-style and Gentzen-style formal representations of theories can be equivalent (also semantically)? If so, which classes of such representations are equivalent and which are not?

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A preliminary answer:
Generally, Gentzen-style theories have no Hilbert-style counterparts. Informal linguistic translations between systems of rules and sets of axioms are not logically innocent and don't provide be themselves any formal equivalence relation.

thank you!

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