A Yoneda lemma-formulation of the univalence axiom

Iosif Petrakis

University of Munich

HoTT/UF 2018
Oxford, 08.07.2018

The question we try to answer

How can one explain UA in more standard mathematical terms?

Previous work on which we are based

Rijke 2012: he gave a type-theoretic formulation of Yoneda lemma and constructed it from Martin-Löf's J-rule and the function extensionality axiom.

Escardó 2015: he took Rijke's type-theoretic formulation of Yoneda lemma as primitive and constructed Martin-Löf's J-rule from it so that its computation rule holds definitionally.

Coquand 2014: he reduced the J-rule to transport and the contractibility of singleton types.

What we do here

We give a Yoneda lemma-formulation (`sY-UA`) of Voevodsky's axiom of univalence (UA) in informal UTT.

Although the computation rules of UA hold propositionally, the computation rules of `sY-UA` hold definitionally.

$$J : \prod_{C:\prod_{x,y:A}\prod_{p:x=_A y}\mathcal{U}} \prod_{c:\prod_{x:A} C(x,x,\mathtt{refl}_x)} \prod_{x,y:A} \prod_{p:x=_A y} C(x,y,p)$$

$$J(C,c,x,x,\mathtt{refl}_x) \equiv c(x), \quad x : A$$

$$\mathtt{LeastRefl} : \prod_{R:A\to A\to\mathcal{U}} \prod_{r:\prod_{x:A} R(x,x)} \prod_{x,y:A} \prod_{p:x=_A y} R(x,y),$$

$$\mathtt{LeastRefl}(R,r,x,x,\mathtt{refl}_x) \equiv r(x), \quad x : A.$$

$$\mathtt{Transport} : \prod_{P:A\to\mathcal{U}} \prod_{x,y:A} \prod_{p:x=_A y} (P(x) \to P(y))$$

$$\mathtt{Transport}(P,x,x,\mathtt{refl}_x) \equiv \mathrm{id}_{P(x)}, \quad x : A.$$

$$j : \prod_{a:A} \prod_{C:\prod_{x:A} \prod_{p:a=_A x} \mathcal{U}} \prod_{c:C(a,\mathtt{refl}_a)} \prod_{x:A} \prod_{p:a=_A x} C(x,p)$$

$$j(a, C, c, a, \mathtt{refl}_a) \equiv c$$

$$\mathtt{leastrefl} : \prod_{a:A} \prod_{R_a:A\to\mathcal{U}} \prod_{r_a:R_a(a)} \prod_{x:A} \prod_{p:a=_A x} R_a(x)$$

$$\mathtt{leastrefl}(a, R_a, r_a, a, \mathtt{refl}_a) \equiv r_a,$$

$$\mathtt{transport} : \prod_{a:A} \prod_{P:A\to\mathcal{U}} \prod_{x:A} \prod_{p:a=_A x} (P(a) \to P(x))$$

$$\mathtt{transport}(a, P, a, \mathtt{refl}_a) \equiv \mathrm{id}_{P(a)}.$$

The J-judgement and the J-computation rule imply the following M-judgement and M-computation rule, respectively,

$$M : \prod_{a,x:A} \prod_{p:a=_A x} (a, \mathtt{refl}_a) =_{E_a} (x, p)$$

$$M(a, a, \mathtt{refl}_a) \equiv \mathtt{refl}_{(a,\mathtt{refl}_a)},$$

where

$$E_a \equiv \sum_{x:\,A} (a =_A x).$$

Similarly we get that the j-judgement and the j-computation rule imply the following m-judgement and m-computation rule, respectively,

$$m : \prod_{a:A} \prod_{u:E_a} (a, \mathtt{refl}_a) =_{E_a} u$$

$$m_a\Big((a, \mathtt{refl}_a)\Big) \equiv \mathtt{refl}_{(a,\mathtt{refl}_a)},$$

where $m_a \equiv m(a)$.

The following two judgements

$$m_a : \prod_{u:E_a} (a, \mathtt{refl}_a) =_{E_a} u$$

$$\mathtt{transport}_a : \prod_{P:A\to\mathcal{U}} \prod_{x:A} \prod_{p:a=_A x} (P(a) \to P(x))$$

imply the judgement

$$j_a : \prod_{C:\prod_{x:A}\prod_{p:a=_A x}\mathcal{U}} \prod_{c:C(a,\mathtt{refl}_a)} \prod_{x:A} \prod_{p:a=_A x} C(x,p)$$

and the same holds for their corresponding computation rules.

[Coquand, 2014] The following judgements and corresponding computation rules are equivalent:

(i) J.
(ii) `Transport` and M.
(iii) `LeastRefl` and M.

Yoneda lemma

$\mathcal{C}$ a locally small category : $\mathrm{Hom}_{\mathcal{C}}(A, B) \equiv \{f \in \mathcal{C}_1 \mid f : A \to B\}$ is a set

$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ the category of contravariant set-valued functors on $\mathcal{C}$

If $C \in \mathcal{C}_0$ and $F \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$, there is an isomorphism

$$\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}}(\mathcal{Y}(C), F) \simeq F(C),$$

which is natural in both F and C, where

$$\mathcal{Y} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

is the **Yoneda embedding** i.e., the functor

$$\mathcal{Y}(C) \equiv \mathrm{Hom}_{\mathcal{C}}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

$$\mathcal{Y}(f : C \to C') \equiv \mathrm{Hom}_{\mathcal{C}}(-, f) : \mathrm{Hom}_{\mathcal{C}}(-, C) \to \mathrm{Hom}_{\mathcal{C}}(-, C')$$

defined post-compositionally.

Through the Yoneda lemma the Yoneda embedding is shown to be an embedding i.e., an injective on objects, faithful, and full functor.

Rijke's type-theoretic interpretation of the Yoneda embedding

$A : \mathcal{U}$ as a locally small category equal to its opposite,
$\mathrm{Hom}(a, b) \equiv a =_A b : \mathcal{U}$

$\mathcal{U}$ is closed under exponentiation, as **Set**

$P : A \to \mathcal{U}$ as an element of $\mathcal{U}^A$, which corresponds to $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$

$$\mathcal{Y} : A \to (A \to \mathcal{U})$$

$$\mathcal{Y}_a : A \to \mathcal{U}$$

$$\mathcal{Y}(a)(x) \equiv x =_A a,$$

$$\mathrm{Hom}(P, Q) \equiv \prod_{x:\,A} \big(P(x) \to Q(x)\big)$$

$$\mathrm{Hom}(\mathcal{Y}(a), P) \equiv \prod_{x:\,A} \big(\mathcal{Y}(a)(x) \to P(x)\big) \equiv \prod_{x:\,A} \big((x =_A a) \to P(x)\big)$$

$$\equiv \prod_{x:\,A} \prod_{p : x =_A a} P(x).$$

Theorem (Yoneda lemma in ITT + Function extensionality (Rijke, 2012))

Let $P : A \to \mathcal{U}$ *and* $a : A$*. There is a pair of quasi-inverses*

$$(j, i) : \mathrm{Hom}(\mathcal{Y}(a), P) \simeq P(a)$$

i.e.,

$$(j \circ i)(u) = u, \quad u : P(a),$$

$$(i \circ j)(\sigma) = \sigma, \quad \sigma : \prod_{x : A} \prod_{p : x =_A a} P(x)$$

such that

$$i(u)(a, \mathtt{refl}_a) \equiv u, \quad u : P(a),$$

$$j(\sigma) \equiv \sigma(a, \mathtt{refl}_a), \quad \sigma : \prod_{x : A} \prod_{p : x =_A a} P(x).$$

Proposition

The $\mathcal{Y}$-judgement implies the introduction rule of the equality type i.e., the inhabitedness of the type $a =_A a$, for every $a \colon A$.

Proof.

If $a \colon A$, and if we consider as P in the type-theoretic Yoneda lemma the type family $\mathcal{Y}(a)$, then

$$\mathrm{Hom}(\mathcal{Y}(a), \mathcal{Y}(a)) \equiv \left(\prod_{x \colon A} \prod_{p : x =_A a} x =_A a \right) \simeq (a =_A a) \equiv \mathcal{Y}(a).$$

The only element of $\mathrm{Hom}(\mathcal{Y}(a), \mathcal{Y}(a))$ we can determine at this point is

$$R \equiv \lambda(x \colon A, p : x =_A a).p$$

and $j(R) : a =_A a$. □

Proposition

The $\mathcal{Y}$-judgement implies the `Transport`*-judgement and the left $\mathcal{Y}$-computation rule implies the* `Transport`*-rule.*

Lemma (Escardó)

If $B : \mathcal{U}$, the $\mathcal{Y}$-judgement and the $\mathcal{Y}$-computation rules imply the following judgement and corresponding computation rules:

$$(j, i) : \left(\prod_{x : A} \prod_{p : x =_A a} B \right) \simeq B$$

$$i(b)(a, \mathtt{refl}_a) \equiv b, \quad b : B,$$

$$j(\sigma) \equiv \sigma(a, \mathtt{refl}_a), \quad \sigma : \prod_{x : A} \prod_{p : x =_A a} B.$$

Moreover, if $b : B$, $x : A$, and $p : x =_A a$, then

$$i(b)(x, p) =_B b.$$

Corollary (Escardó)

The $\mathcal{Y}$-judgement with the $\mathcal{Y}$-computation rules imply the M-judgement.

The next theorem is shown without the use of function extensionality.

Theorem (Escardó, 2015)

The J-judgement and the J-computation rule follow from the $\mathcal{Y}$-judgement and the $\mathcal{Y}$-computation rules.

The univalence axiom asserts that the function $\mathtt{IdtoEqv}(X) : X =_{\mathcal{U}} A \to X \simeq_{\mathcal{U}} A$ is an equivalence with quasi-inverse the function $\mathtt{ua}(X) : X \simeq_{\mathcal{U}} A \to X =_{\mathcal{U}} A$.

Voevodsky's Axiom of Univalence ($\mathtt{UA}$): There are the following $\mathtt{ua}$-judgement and the right and left $\mathtt{ua}$-computation rules, respectively,

$$\mathtt{ua} : \prod_{X:\mathcal{U}} \prod_{e:X \simeq_{\mathcal{U}} A} X =_{\mathcal{U}} A$$

$$\mathtt{ua}(X, \mathtt{IdtoEqv}(X, p)) = p, \quad p : X =_{\mathcal{U}} A,$$

$$[\mathtt{IdtoEqv}(X, \mathtt{ua}(e))]^*(x) = e^*(x), \quad x : X.$$

$$\mathtt{IdtoEqv}(\mathtt{ua}(f), x) = f(x),$$

where the equivalence e is "identified" with $f \equiv e^*$

$$\mathtt{ua}(A, (\mathrm{id}_A, e_A)) = \mathtt{refl}_A.$$

The "categorical" interpretation

$\mathcal{U}$ as a locally small category equal to its opposite,
$\mathrm{Hom}(A, B) \equiv A \simeq_{\mathcal{U}} B : \mathcal{U}$

$\mathcal{U}'$, the next universe to $\mathcal{U}$, as **Set**

$P : \mathcal{U} \to \mathcal{U}'$ as an element of $\mathcal{U}'^{\mathcal{U}}$, which corresponds to $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$

$$\mathcal{E} : \mathcal{U} \to (\mathcal{U} \to \mathcal{U}')$$

$$\mathcal{E}_A(X) \equiv X \simeq_{\mathcal{U}} A,$$

$$e : A \simeq_{\mathcal{U}} B$$

$$\mathcal{E}(e) : \mathrm{Hom}(\mathcal{E}_A, \mathcal{E}_B) \equiv \prod_{X:\mathcal{U}} \prod_{e':X \simeq_{\mathcal{U}} A} X \simeq_{\mathcal{U}} B$$

$$\mathcal{E}(e) \equiv \lambda(X : \mathcal{U}, e' : X \simeq_{\mathcal{U}} A).e \circ e'.$$

Yoneda-version of the univalence axiom (`Y-UA`): Let $P : \mathcal{U} \to \mathcal{U}'$ and $A : \mathcal{U}$. There is a pair of quasi-inverses

$$(j, i) : \mathsf{Hom}(\mathcal{E}_A, P) \simeq P(A)$$

i.e., there are the following i-judgment and j-judgment:

$$i : P(A) \to \prod_{X:\mathcal{U}} \prod_{e:X \simeq_{\mathcal{U}} A} P(X)$$

$$j : \left(\prod_{X:\mathcal{U}} \prod_{e:X \simeq_{\mathcal{U}} A} P(X) \right) \to P(A)$$

with the following i-computation rule and j-computation rule:

$$i(u)(A, (\mathrm{id}_A, e_A)) \equiv u, \quad u : P(A),$$

$$j(\sigma) \equiv \sigma(A, (\mathrm{id}_A, e_A)), \quad \sigma : \mathsf{Hom}(\mathcal{E}_A, P).$$

Proposition

The i-judgement of Y-UA *implies the* ua*-judgement i.e., there is*

$$\mathtt{ua}' : \prod_{X:\mathcal{U}} \prod_{e:X \simeq_{\mathcal{U}} A} X =_{\mathcal{U}} A,$$

$$\mathtt{ua}'(A, (\mathrm{id}_A, e_A)) \equiv \mathtt{refl}_A.$$

Proof.

Let $P : \mathcal{U} \to \mathcal{U}'$ defined by $P(X) \equiv X =_{\mathcal{U}} A$. Since

$$i : A =_{\mathcal{U}} A \to \prod_{X:\mathcal{U}} \prod_{e:X \simeq_{\mathcal{U}} A} X =_{\mathcal{U}} A,$$

$$\mathtt{ua}' \equiv \lambda(X : \mathcal{U}, e : X \simeq_{\mathcal{U}} A).i(\mathtt{refl}_A)(X, e),$$

hence

$$\mathtt{ua}'(A, (\mathrm{id}_A, e_A)) \equiv i(\mathtt{refl}_A)(A, (\mathrm{id}_A, e_A)) \equiv \mathtt{refl}_A.$$

Proposition

If $X : \mathcal{U}$ *and* $p : X =_{\mathcal{U}} A$*, then*

$$\mathtt{ua}'(X, \mathtt{IdtoEqv}(X, p)) = p.$$

Proof.

Define $C(X, p) \equiv \mathtt{ua}'(X, \mathtt{IdtoEqv}(X, p)) = p$. Since

$$\begin{aligned} C(A, \mathtt{refl}_A) &\equiv \mathtt{ua}'(A, \mathtt{IdtoEqv}(A, \mathtt{refl}_A)) = \mathtt{refl}_A \\ &\equiv \mathtt{ua}'(A, (\mathrm{id}_A, e_A)) = \mathtt{refl}_A \\ &\equiv \mathtt{refl}_A = \mathtt{refl}_A, \end{aligned}$$

we use the j_A-judgment. □

Proposition

The $\mathtt{ua}$*-judgement implies the i-judgement of* $\mathtt{Y-UA}$ *i.e., there is*

$$i' : P(A) \to \prod_{X:\mathcal{U}} \prod_{e:X \simeq_{\mathcal{U}} A} P(X),$$

and moreover

$$i'(u)(A, (\mathrm{id}_A, e_A)) = u, \quad u : P(A).$$

Proof.

Let $u : P(A)$. Since $\mathtt{ua}(X, e) : X =_{\mathcal{U}} A$, we get $\mathtt{ua}(X, e)^{-1} : A =_{\mathcal{U}} X$, and consequently we have that

$$\left[\mathtt{ua}(X, e)^{-1}\right]_*^P : P(A) \to P(X).$$

We define

$$i'(u) \equiv \lambda(X : \mathcal{U}, e : X \simeq_{\mathcal{U}} A).\left[\mathtt{ua}(X, e)^{-1}\right]_*^P(u).$$

Thus,

$$\begin{aligned} i'(u)(A, (\mathrm{id}_A, e_A)) &\equiv \left[\mathtt{ua}(A, (\mathrm{id}_A, e_A))^{-1}\right]_*^P(u) \\ &= \left(\mathtt{refl}_A^{-1}\right)_*^P(u) \\ &\equiv \left(\mathtt{refl}_A\right)_*^P(u) \\ &\equiv \mathrm{id}_{P(A)}(u) \\ &\equiv u. \end{aligned}$$

Our aim is to get from a strong Yoneda version of the axiom of univalence the J-judgement that corresponds to it equipped with a computation rule that involves judgemental equality.

Let $A, B : \mathcal{U}$ and $Q : A \to B \to \mathcal{U}'$ a type family over A and B (or a relation on A, B). If

$$F, G : \prod_{x:A} \prod_{y:B} Q(x, y),$$

we say that F, G are **homotopic**, $F \approx B$, if there is

$$H : F \approx B \equiv \prod_{x:A} \prod_{y:B} F(x, y) =_{Q(x,y)} G(x, y).$$

Proposition

Let $A : \mathcal{U}$ *and* $P : \mathcal{U} \to \mathcal{U}'$. *If we fix some*

$$\sigma : \mathrm{Hom}(\mathcal{E}_A, P) \equiv \prod_{X:\mathcal{U}} \prod_{f:X\simeq A} P(X),$$

there is a term

$$\mathtt{Happly}_{\mathcal{E},\sigma} : \prod_{\tau\in\mathrm{Hom}(\mathcal{E}_A,P)} \prod_{p:\tau=\sigma} \tau \approx \sigma \equiv$$

$$\equiv \prod_{\tau\in\mathrm{Hom}(\mathcal{E}_A,P)} \prod_{p:\tau=\sigma} \left(\prod_{X:\mathcal{U}} \prod_{f:X\simeq A} \tau(X,f) =_{P(X)} \sigma(X,f) \right)$$

$$\mathtt{Happly}_{\mathcal{E},\sigma}(\sigma, \mathtt{refl}_\sigma) \equiv \lambda(X:\mathcal{U}, f : X \simeq A).\mathtt{refl}_{\sigma(X,f)},$$

Proof.

If $C(\tau, p) \equiv \tau \approx \sigma$, then $C(\sigma, \mathtt{refl}_\sigma) \equiv \sigma \approx \sigma$ and $\lambda(X:\mathcal{U}, f : X \simeq A).\mathtt{refl}_{\sigma(X,f)} : C(\sigma, \mathtt{refl}_\sigma)$. □

Strong Yoneda-version of the univalence axiom (sY-UA): Let $A:\mathcal{U}$ and $P:\mathcal{U}\to\mathcal{U}'$. There is a pair of quasi-inverses

$$i : P(A) \to \prod_{X:\mathcal{U}} \prod_{f:X\simeq A} P(X),$$

$$j : \left(\prod_{X:\mathcal{U}} \prod_{f:X\simeq A} P(X)\right) \to P(A),$$

equipped with the following i and j-computation rule:

$$i(u)(A,\mathrm{id}_A) \equiv u, \quad u : P(A),$$

$$j(\sigma) \equiv \sigma(A,\mathrm{id}_A), \quad \sigma : \mathsf{Hom}(\mathcal{E}_A, P).$$

Moreover, there are terms

$$G : i\circ j \sim \mathrm{id}_{\mathsf{Hom}(\mathcal{E}_A,P)} \equiv \prod_{\sigma\in\mathsf{Hom}(\mathcal{E}_A,P)} i(j(\sigma)) = \sigma,$$

$$H : j\circ i \sim \mathrm{id}_{P(A)} \equiv \prod_{u:P(A)} j(i(u)) = u,$$

$$\mathtt{Happly}_{\mathcal{E},\sigma}\big(i(j(\sigma)), G(\sigma)\big)(A,\mathrm{id}_A) \equiv \mathtt{refl}_{\sigma(A,\mathrm{id}_A)}, \quad \sigma : \mathsf{Hom}(\mathcal{E}_A, P),$$

$$H(u) \equiv \mathtt{refl}_u, \quad u : P(A).$$

Since

$$G(\sigma) : i(j(\sigma)) = \sigma,$$

we have that

$$\mathtt{Happly}_{\mathcal{E},\sigma}\big(i(j(\sigma)), G(\sigma)\big) : \prod_{X:\mathcal{U}} \prod_{f:X\simeq A} i(j(\sigma))(X,f) =_{P(X)} \sigma(X,f),$$

$$\mathtt{Happly}_{\mathcal{E},\sigma}\big(i(j(\sigma)), G(\sigma)\big)(A, \mathrm{id}_A) : i(j(\sigma))(A, \mathrm{id}_A) =_{P(A)} \sigma(A, \mathrm{id}_A).$$

By the j, i-computation rules we have that

$$i(j(\sigma))(A, \mathrm{id}_A) \equiv i(\sigma(A, \mathrm{id}_A))(A, \mathrm{id}_A) \equiv \sigma(A, \mathrm{id}_A),$$

therefore

$$\mathtt{Happly}_{\mathcal{E},\sigma}\big(i(j(\sigma)), G(\sigma)\big)(A, \mathrm{id}_A) : \sigma(A, \mathrm{id}_A) =_{P(A)} \sigma(A, \mathrm{id}_A).$$

Similarly, if $u : P(A)$,

$$H(u) : j(i(u)) = u,$$

and since

$$j(i(u)) \equiv i(u)(A, \mathrm{id}_A) \equiv u,$$

we get

$$H(u) : u =_{P(A)} u.$$

Lemma

If $B : \mathcal{U}$, the strong Yoneda-judgements and the corresponding computation rules imply the following judgement and computation rules:

$$(j_B, i_B) : \left(\prod_{X:\mathcal{U}} \prod_{f:X \simeq A} B \right) \simeq B$$

$$i_B : B \to \prod_{X:\mathcal{U}} \prod_{f:X \simeq A} B,$$

$$j_B : \left(\prod_{X:\mathcal{U}} \prod_{f:X \simeq A} B \right) \to B,$$

$$i_B(b)(A, \mathrm{id}_A) \equiv b, \quad b : B,$$

$$j_B(\sigma) \equiv \sigma(A, \mathrm{id}_A), \quad \sigma : \prod_{X:\mathcal{U}} \prod_{f:X \simeq A} B,$$

$$G_B : \prod_{\sigma \in \mathrm{Hom}(\mathcal{E}_A, B)} i_B(j_B(\sigma)) = \sigma,$$

Lemma

$$H_B : \prod_{b:B} j(i(b)) = b,$$

$$\mathtt{Happly}_{\mathcal{E},\sigma}\Big(i_B(j_B(\sigma)), G_B(\sigma)\Big)(A, \mathrm{id}_A) \equiv \mathtt{refl}_{\sigma(A,\mathrm{id}_A)}, \quad \sigma : \mathrm{Hom}(\mathcal{E}_A, B),$$

$$H_B(b) \equiv \mathtt{refl}_b, \quad b : B.$$

Moreover, if $b : B$, $X : \mathcal{U}$ and $f : X \simeq A$, then, if

$$[\sigma_b \equiv \lambda(X : \mathcal{U}, f : X \simeq A).b] : \prod_{X:\mathcal{U}} \prod_{f:X\simeq A} B,$$

we have that

$$\mathtt{Happly}_{\mathcal{E},\sigma_b}\Big(i_B(j_B(\sigma_b)), G_B(\sigma_b)\Big)(X, f) : \big[i_B(b)(X, f) =_B b\big],$$

$$\mathtt{Happly}_{\mathcal{E},\sigma_b}\Big(i_B(j_B(\sigma_b)), G_B(\sigma_b)\Big)(A, \mathrm{id}_A) \equiv \mathtt{refl}_b.$$

Corollary

If

$$E_A \equiv \sum_{X:\mathcal{U}} X \simeq A,$$

the judgements and computational rules of the strong Yoneda-version of UA imply the following M_e-judgement and M_e-computation rule:

$$M_e : \prod_{X:\mathcal{U}} \prod_{f:X\simeq A} (X, f) =_{E_A} (A, \mathrm{id}_A),$$

$$M_e(A, \mathrm{id}_A) \equiv \mathtt{refl}_{(A,\mathrm{id}_A)}.$$

We call the following judgment and computation rule

$$J_e : \prod_{C:\prod_{X:\mathcal{U}}\prod_{f:X\simeq A}\mathcal{U}} \prod_{c:C(A,\mathrm{id}_A)} \left(\prod_{X:\mathcal{U}} \prod_{f:X\simeq A} C(X,f) \right)$$

$$J_e(C, c, A, \mathrm{id}_A) \equiv c$$

the Eq−J-judgement and the Eq−J-computation rule, respectively.

Theorem

The judgements and computational rules of the strong Yoneda-version of UA imply the Eq−J*-judgement and the* Eq−J*-computation rule.*

Proof

We fix $C : \prod_{X:\mathcal{U}} \prod_{f:X\simeq A} \mathcal{U}$ and $c \in C(A, \mathrm{id}_A)$. Let $E_A \equiv \sum_{X:\mathcal{U}} X \simeq A$, and $P : E_A \to \mathcal{U}$, defined by

$$P((X, f)) \equiv C(X, f),$$

for every $X : \mathcal{U}$ and $f : X \simeq A$. By Corollary

$$M_e(X, f) : (X, f) =_{E_A} (A, \mathrm{id}_A),$$

hence

$$M_e(X, f)^{-1} : (A, \mathrm{id}_A) =_{E_A} (X, f).$$

Consequently

$$\left[M_e(X, f)^{-1}\right]_*^P : P((A, \mathrm{id}_A)) \to P((X, f)) \equiv C(A, \mathrm{id}_A) \to C(X, f).$$

We define

$$J_e(C, c, X, f) \equiv \left[M_e(X, f)^{-1}\right]_*^P(c).$$

By the corollary we get

$$
\begin{aligned}
J_e(C, c, A, \mathrm{id}_A) &\equiv \left[M_e(A, \mathrm{id}_A)^{-1}\right]_*^P(c) \\
&\equiv \left[(\mathtt{refl}_{(A,\mathrm{id}_A)})^{-1}\right]_*^P(c) \\
&\equiv \left[\mathtt{refl}_{(A,\mathrm{id}_A)}\right]_*^P(c) \\
&\equiv \mathrm{id}_{P((A,\mathrm{id}_A))}(c) \\
&\equiv \mathrm{id}_{C(A,\mathrm{id}_A)}(c) \\
&\equiv c.
\end{aligned}
$$

Corollary

If $f : X \simeq A$*, then* $\mathtt{IdtoEqv}(X, \mathtt{ua}'(X, f)) = f$.

Proof.

We define $C(X, f) \equiv \mathtt{IdtoEqv}(X, \mathtt{ua}'(X, f)) = f$. Since

$$\begin{aligned} C(A, \mathrm{id}_A) &\equiv \mathtt{IdtoEqv}(A, \mathtt{ua}'(A, \mathrm{id}_A)) = \mathrm{id}_A \\ &\equiv \mathtt{IdtoEqv}(A, \mathtt{refl}_A) = \mathrm{id}_A \\ &\equiv \mathrm{id}_A = \mathrm{id}_A, \end{aligned}$$

we have that $\mathtt{refl}_{\mathrm{id}_A} : C(A, \mathrm{id}_A)$, and we use the Theorem. □

Concluding remarks

- The proximity of UA to the J-rule is shown here also from the categorical point of view. Both admit a Yoneda-lemma formulation.
- The strong Yoneda lemma-formulation of univalence supports the definitional approach to the computational rules associated to the judgements of type theory. It is also used to construct Voevodsky's formulation of univalence.
- We need to check `sY−UA` in models of UTT.

- T. Coquand: A remark on singleton types, manuscript, 2014.
- M. Escardó: Using Yoneda rather than J to present the identity type, Agda file, in http://www.cs.bham.ac.uk/∼mhe/yoneda/yoneda.html
- I. Petrakis: A Yoneda lemma-formulation of the univalence axiom, preprint, 2018.
- E. Rijke: *Homotopy Type Theory*, Master Thesis, Utrecht University 2012.
- The Univalent Foundations Program: *Homotopy Type Theory: Univalent Foundations of Mathematics*, Institute for Advanced Study, Princeton, 2013.