Internalizing Presheaf Semantics: Charting the Design Space

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LICS '18
Oxford, UK
July 7, 2018

Presheaf semantics can model:

- **Parametricity** (preservation of **relations**),
- **HoTT** (preservation of **equivalences**),
- **Directed TT** (preservation of **homomorphisms**).

Operators for **cheap proofs** of **free preservation theorems**?

- Cubical TT: Glue
- NVD17, ND18: Glue, Weld
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Universal Type Extension Operators (Glue, Weld)

Final extension of (T, f)

$$\frac{\Gamma \vdash A\ \textbf{type} \qquad \Gamma \vdash P\ \textbf{prop} \qquad \Gamma, P \vdash T\ \textbf{type} \qquad \Gamma, P \vdash f : T \to A}{\Gamma \vdash \textbf{Glue}\,\{A \leftarrow (P\,?\,T, f)\}\ \textbf{type} \qquad \Gamma, P \vdash \textbf{Glue}\,\{A \leftarrow (P\,?\,T, f)\} = T}$$

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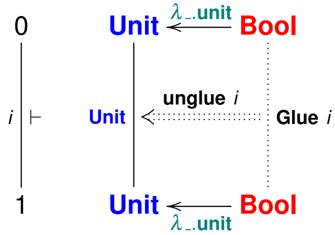

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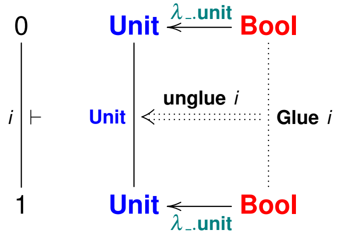

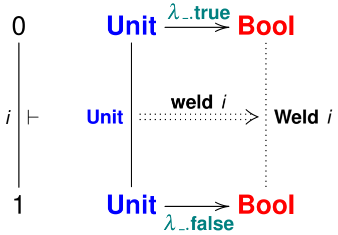

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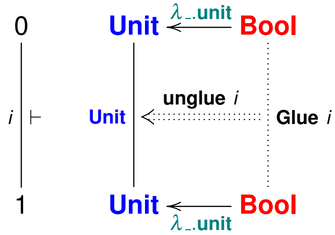
Theorem (Cubical TT)

$$\frac{A\ \text{Kan} \qquad T\ \text{Kan} \qquad f\ \text{equivalence}}{\textbf{Glue}\ \text{Kan}}$$



As (co-)inductive types

Final extension of (T, f)

record $G := \textbf{Glue}\,\{A \leftarrow (P\,?\,T, f)\}$
where

- **unglue** : $G \to A$
- **reduce** : $G \to (\_ : P) \to T$
- **coh** : $(g : G) \to (\_ : P) \to$
  $f\ (\textbf{reduce}\ g\ \_) \equiv_A \textbf{unglue}\ g$

G extends T
unglue extends f
reduce is id_T
coh is **refl**

Initial extension of (T, f)

HIT $W := \textbf{Weld}\,\{A \to (P\,?\,T, f)\}$
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- **weld** : $A \to W$
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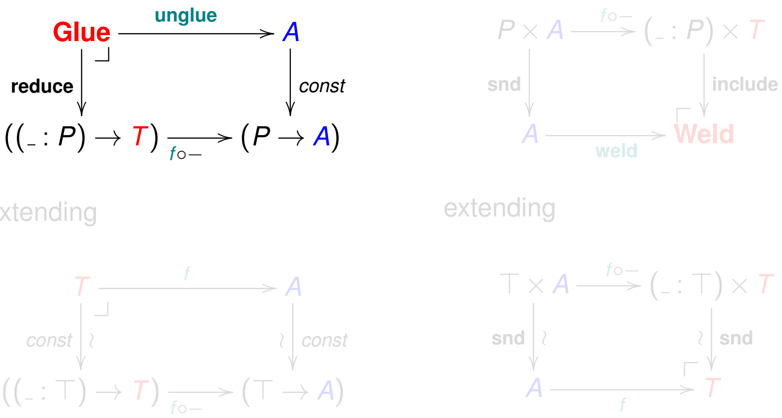

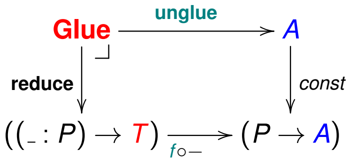

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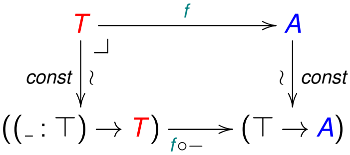

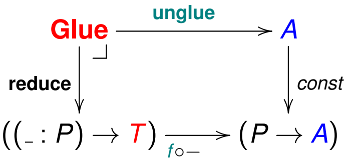

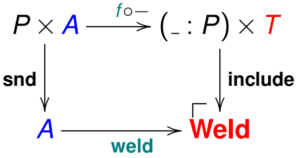

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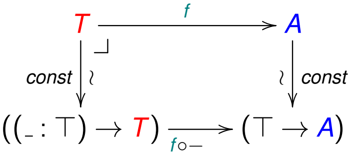

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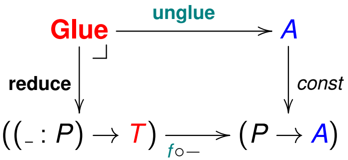

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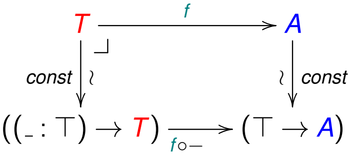

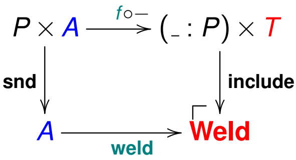

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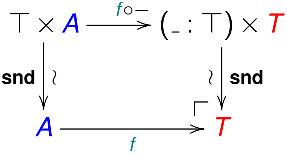

A milling cutter

NL: frees

FR: fraise

DE: Fräser

PL: frez

JP: furaisu

CMN: xǐdāo

If base category has products;
for any **shape** $\mathbb{I}$:

$$\begin{array}{ccc} \Pi\mathbb{I}P \times \Pi\mathbb{I}A & \longrightarrow & (\_ : \Pi\mathbb{I}P) \times \Pi\mathbb{I}T \\ \downarrow & & \downarrow \\ \Pi\mathbb{I}A & \longrightarrow & \Pi\mathbb{I}\mathbf{Weld} \end{array}$$

$$\mathbf{mill} :$$
$$(\Pi i.\mathbf{Weld}\,\{A\,i \to (P\,i\,?\,T\,i, f\,i)\})$$
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$$\mathbf{Weld}\,\{\Pi i.A\,i \to (\forall i.P\,i\,?\,\Pi i.T\,i, f \circ -)\}$$

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Debating nomenclature

Glue	Weld
glue	weld
unglue	
Glue	Coglue
glue	co**un**glue
unglue	
FExt	IExt
?	?
?	

Different name in Cubical TT?

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Glue/Weld: Summary

Universal type extension operators (Glue/Weld):

- Exist in any presheaf model,
- Internalize nothing about the **particular** model.

Combine with something else:

- Box filling (Cubical TT)
- Modalities & identity extension lemma [NVD17]
- **mill** (identifies **shape** types for particular model)

Boundary-Filling Operators
(Φ, Ψ)

Generalized from:
Bernardy, Coquand, Moulin (2015)
Moulin's PhD (2016)

Boundaries

Definition (Boundary)

For any shape $\mathbb{I} \in \mathcal{C}$,
the **boundary** is the greatest strict subobject $\partial\mathbb{I} \subsetneq \mathbf{y}\mathbb{I} \in \widehat{\mathcal{C}}$.

Theorem

$$(\mathbf{y}\mathbb{U} \to \partial\mathbb{I}) \cong (\mathbb{U} \to \mathbb{I}) \setminus \{\text{split epis}\}.$$

Note:

$\varphi : \mathbb{U} \to \mathbb{V}$ split epi $\Leftrightarrow$ $\mathbf{y}\varphi : \mathbf{y}\mathbb{U} \to \mathbf{y}\mathbb{V}$ epi

$\varphi : \mathbb{U} \to \mathbb{V}$ mono $\Leftrightarrow$ $\mathbf{y}\varphi : \mathbf{y}\mathbb{U} \to \mathbf{y}\mathbb{V}$ mono

Example

In Cubical TT: $\partial\mathbb{I} \cong \mathbf{Bool}$.

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Fillers

$$\frac{\Gamma, i : \mathbb{I} \vdash A\,\textbf{type} \qquad \Gamma, i : \partial\mathbb{I} \vdash a : A}{\Gamma \vdash \textbf{Filler}_{i.A}\,(i.a)\,\textbf{type}}$$

$$\textbf{Filler}_{i.A}\,(i.a) \cong (f : (i : \mathbb{I}) \to A) \times ((i : \partial\mathbb{I}) \to f\ i =_A a)$$

Example (Cubical TT)

$\textbf{Filler}_{i.A}\,(i.a) = \textbf{Path}_{i.A}(a[0/i], a[1/i])$.

Boundary-filling

For any shape $\mathbb{I}$:

$$\frac{\Gamma \vdash f_\partial : (i : \partial\mathbb{I}) \multimap (x : A\ i) \to B\ i\ x \qquad \Gamma \vdash h : (\xi : (i : \mathbb{I}) \multimap A\ i) \to \mathbf{Filler}_{i.B\ i\ (\xi\ i)}\ (f_\partial\ i\ (\xi\ i))}{\Gamma \vdash \Phi(f_\partial, h) : (i : \mathbb{I}) \multimap (x : A\ i) \to B\ i\ x}$$

$$\Phi(f_\partial, h)|_{\partial\mathbb{I}} = f_\partial, \qquad \Phi(f_\partial, h)\ i\ a = h\ (\lambda i.a)\ i$$

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Compare: funext and univalence

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Problem with diagonals

How to build

$f : (i, j : \mathbb{I}) \multimap (x : A\ i\ j) \to B\ i\ j\ x$?

$f =$

$\Phi^2(f_{00}, f_{01}, f_{10}, f_{11}, h_0, h_1, k_0, k_1, w)$

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w

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Solution:

Base category: $\mathbb{I} \not\to \mathbb{I} * \mathbb{I}$

Separated product: (cf. nom. sets)

$[\![\Gamma, i : \mathbb{I}]\!] = [\![\Gamma]\!] * \mathbf{y}\mathbb{I}$

"Linear" application:

$$\frac{\Gamma \vdash f : (i : \mathbb{I}) \multimap A\ i}{\Gamma, i : \mathbb{I} \vdash f\ i : A\ i}$$

Incorporating Φ (J-P. Bernardy):

$$\frac{\Gamma, i : \partial\mathbb{I}, \Delta \vdash a_\partial : A\ i \qquad \Gamma, (i : \mathbb{I}) \multimap \Delta \vdash h : \mathbf{Filler}_{i.A\ i[\delta\ i/\delta]}\ (a_\partial[\delta\ i/\delta])}{\Gamma, i : \mathbb{I}, \Delta \vdash h\ i : A\ i}$$

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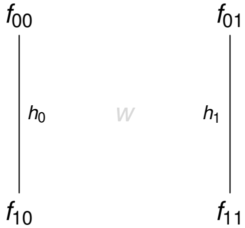

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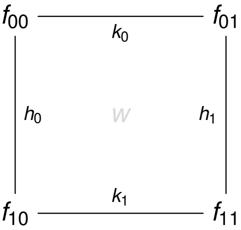

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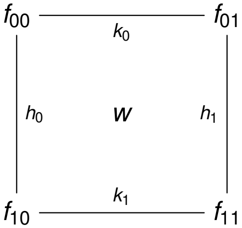

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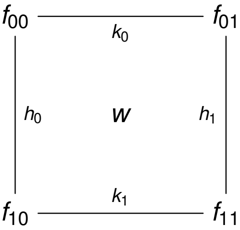

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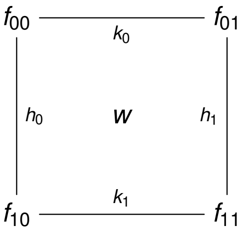

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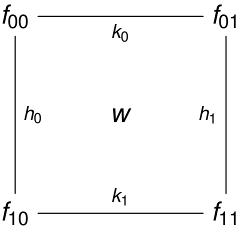

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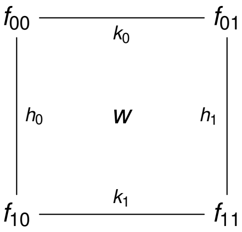

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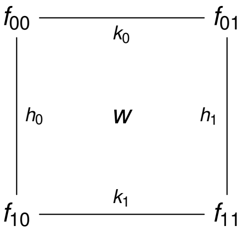

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Exchange only works one way.

$$(\Gamma, a : A, i : \mathbb{I}, \Delta)$$
$$\cong$$
$$(\Gamma, i : \mathbb{I}, a : A, i \# a, \Delta)$$
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"a does not vary with i."

Freshness predicate

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"*a* does not vary with *i*."

Details on semantics

Definition (Suitable base category)

- sym. **monoidal** with **terminal unit**,
- projection $(\mathbb{U} * \mathbb{V}) \to \mathbb{V}$ cartesian on monos,
- generalized Reedy w.r.t. **split epis** and **monos** (can be relaxed)
- all $\mathbb{U} \to \top$ split epi, equiv.: $\partial\top = \varnothing$

Definition

$\mathcal{C}//\mathbb{U}$: split epi slices.

Theorem

$(\mathbb{U} * -, \pi_1) : \mathcal{C} \to \mathcal{C}//\mathbb{U}$ is **faithful**.

Definition (Diagonal-free)

$(\mathbb{U} * -, \pi_1) : \mathcal{C} \to \mathcal{C}//\mathbb{U}$ is **full**.
Rules out
$\delta : (\mathbb{U} * \top, \pi_1) \not\to (\mathbb{U} * \mathbb{U}, \pi_1)$

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Definition (Cartesian)

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Requirements

	Glue	Weld	mill	Φ	Ψ	$\#$
mon. base cat.			•	(•)	(•)	(•)
suit. base cat.				•	•	•
cartesian						
diag.-free				•?	•?	•
conn.-free				•	•[1]	•

[1] With connections: Ψ is sound but underspecified.

Results

Theorem

$\Psi, \Phi,$ colimit systems $\models$ **Glue**, **Weld**, **mill**
(where P ranges only over a shape $\mathbb{U}$)

Sketch of proof: By induction on Reedy-degree of $\mathbb{U}$

- Define **Glue**/**Weld**/**mill** on $\partial\mathbb{U}$
  $\partial\mathbb{U} = \mathbf{colim}_i \mathbb{V}_i$ ($\deg \mathbb{V}_i < \deg \mathbb{U}$)
  IH: defined on $\mathbb{V}_i$
  Colimit system: paste together for $\partial\mathbb{U}$
- Fill the boundary using Φ/Ψ. □

Theorem

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Theorem

$$\textbf{Weld}, \textbf{mill}, \# \models \Psi$$

Sketch of proof:

$\Psi(A_\partial, P) : \mathbb{I} \multimap \mathcal{U}$

$\Psi(A_\partial, P)\ i = \textbf{Weld}\,\{A_\Sigma\ i \to (i \in \partial\mathbb{I}\ ?\ A_\partial\ i, f\ i)\}$

$((i : \mathbb{I}) \multimap \Psi(A_\partial, P)\ i)$

$=_{\text{def}} ((i : \mathbb{I}) \multimap \textbf{Weld}\,\{A_\Sigma\ i \to (i \in \partial\mathbb{I}\ ?\ A_\partial\ i, f\ i)\})$

$\cong_{\textbf{mill}} \textbf{Weld}\,\{((i : \mathbb{I}) \multimap A_\Sigma\ i) \to (\bot\ ?\ \lightning, \lightning)\}$

$\cong_{\textbf{ind}_{\textbf{Weld}}} ((i : \mathbb{I}) \multimap A_\Sigma\ i)$

$\cong_{\text{wanted}} (a_\partial : (j : \partial\mathbb{I}) \multimap A_\partial\ j) \times (p : P\ a_\partial)$

$A_\Sigma : \mathbb{I} \multimap \mathcal{U}$

$A_\Sigma\ i = (a_\partial : (j : \partial\mathbb{I}) \multimap A_\partial\ j) \times (p : P\ a_\partial) \times ((a_\partial, p) \# i)$

$\Rightarrow \quad ((i : \mathbb{I}) \multimap A_\Sigma\ i) \cong (a_\partial : (j : \partial\mathbb{I}) \multimap A_\partial\ j) \times (p : P\ a_\partial)$

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Theorem

$$\mathbf{Weld}, \mathbf{mill}, \# \models \Psi$$

Sketch of proof:

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Theorem

$$\textbf{Glue}, \textbf{Weld}, \textbf{mill}, \Psi, \# \not\models \Phi$$

Sketch of proof: Pick fully faithful functor $I : \mathcal{D} \to \mathcal{C}$.
$\widehat{\mathcal{C}} \models \textbf{Glue}_{\mathcal{D}}, \textbf{Weld}_{\mathcal{D}}, \textbf{mill}_{\mathcal{D}}, \Psi_{\mathcal{D}}, \#_{\mathcal{D}}$,
$\widehat{\mathcal{C}} \not\models \Phi_{\mathcal{D}}$ (in general, e.g. ∇ : **Cube** $\to$ **BPCube**)
because $\Phi_{\mathcal{D}}(f_\partial, h)$ has no action on $\mathbb{U}$-cells for $\mathbb{U} \in \mathcal{C} \setminus I(\mathcal{D})$. □

$$[\![-]\!] : \{\text{System F types}\} \to \{\text{MLTT types}\}$$

Theorem

$\Phi_{\mathbf{Cube}}, \Psi_{\mathbf{Cube}} \models$ Every term $t : [\![T]\!]$ is parametric.

Sketch of proof:
use Ψ to convert $(A_0, A_1, \bar{A})$ to $A : \mathbb{I} \multimap \mathcal{U}$,
use Φ to convert $(f_0, f_1, \bar{f})$ to $f : (i : \mathbb{I}) \multimap A\ i \to B\ i$. □

Theorem

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Proof: BPCube models LHS, not RHS. □

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Conclusion

Cartesian Φ and Ψ would be best. (Working on it.)
Alas: they don't play well with connections.

Thanks!

Questions?